Results on Exponential Stability of Impulsive Functional Differential Equation with Infinite or Finite Delays

Kanwalpreet Kaur[1*], S.K.Srivastava[2]

[1]Department of Mathematics, IKGPTU, Jalandhar, India
*Corresponding Author E-mail: kanwal.2015@rediffmail.com

[2]Department of Mathematics, BCET, Gurdaspur, India
E-mail: sks64_bcet@yahoo.co.in

Abstract: This paper studies the exponential stability of impulsive functional differential equation with infinite delays or finite delays by using Razumikhin technique and Lyapunov functions. The obtained results improve some of the earlier findings and are suitable for many applications.

Keywords: Impulsive delay differential system, Lyapunov function, Razumikhin technique, Time delays.

I. INTRODUCTION

Impulsive delay differential equations have attracted many researchers' attention due to their wide applications in many fields such as control technology, drug administration and threshold theory in biology etc. [4-6, 7]. In recent years, impulsive differential systems have been researched intensively there is enough work is done in the qualitative theory of functional differential equations. However, there is not much has been done in the field of impulsive functional differential equations. So by using Lyapunov functions and Razumikhin techniques, some criterion on Razumikhin type theorems on stability is obtained for a class of impulsive functional differential equations with finite delays.

Also time delay exists in various fields in our society; due to this reason the systems with time delay have received major attention in recent years. Because of this reason the systems with infinite delay deserve study because they portray a type of system existing in the real world [1-3,8] .In this paper, we consider impulsive infinite delay differential equations, by using Lyapunov functions and the Razumikhin technique, we obtain some results. The obtained results improve and complement some recent work.

This paper is organized as follows. In Section II, we introduce some basic definitions and notations. In Section III, we get some criteria for stability of impulsive differential equations with finite or infinite delay.

II. PRELIMINARIES

Consider the impulsive functional differential system

$$\begin{cases} x'(t) = f(t, x_t), & t \neq t_k, t \geq t_0 \\ \boldsymbol{x}(\boldsymbol{t_k}) = \boldsymbol{x}(\boldsymbol{t_k^-}) + \boldsymbol{I_k}(\boldsymbol{x}(\boldsymbol{t_k^-})), & \boldsymbol{k} \in \boldsymbol{N} \end{cases} \qquad (1)$$

$x_\sigma = \psi(s)$, Where $s \in [\alpha, 0]$

Where $\sigma \geq t_0 \geq 0, x \in R^n$ and $\geq t_0 > 0 > \alpha \geq -\infty$.$f: [0, \infty) \times C \rightarrow R^n$, where C is an open set in PC($[\alpha, 0], R^n$) where PC($[\alpha, 0], R^n$) ={ψ : [α,0] → R^n is continuous everywhere except at a finite number of points t_k ; at which $\psi(t_k^+), \psi(t_k^-)$ exist and $\psi(t_k^+) = \psi(t_k)$,the impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots\cdots < t_k \rightarrow \infty$ as k→ ∞, $sup_{k \in Z_+}\{t_k - t_{k-1}\} < \infty$ and x' denotes the right-hand derivative of x, $\psi \in$ C;. For each t ≥ $t_0, x_t \in C$ is defined by $x_t(s) = x(t + s), s \in [\alpha, 0]$. Define PCB(t)={ $x_t \in C: x_t$ is bounded}.For $\psi \in PCB(t)$ the norm of ψ is defined by $||\psi|| = sup_{\alpha \leq \theta \leq 0}|\psi(\theta)|$.

$K = \{a \in C(R^+, R^+)|a(0) = 0 \text{ and } a(s) > 0 \text{ for } s > 0 \text{ and a is strictly increasing in s}\}$

Lemma1: The initial problem (1) have a unique solution $x(t) = x(t, \sigma, \psi)$ if the following hypotheses hold:

(i)f:[$t_{k-1}, t_k) \times C \rightarrow R^n, k \in Z^+$ is continuous and for all k $\in Z_+$ and for any $\psi \in C$,the limit $lim_{(t,x)\rightarrow(t_k^-,\psi)} f(t, x) = f(t_k^-, \psi)$ exists.

(ii)$f(t, \psi)$ is Lipschitzian in ψ in each compact set in C.

(iii) $I_k(t, x): [t_0, \infty) \times R^n \rightarrow R^n$ is continuous and for any $\rho > 0$,there exists a $\rho_1 \epsilon (0, \rho)$ such that x ∈ ρ_1 implies that $x + I_k(t_k, x) \in S(\rho) = \{x: |x| < \rho, x \epsilon R^n\}$.

(iv)For any $\psi \in C, f(t, \psi(.)) \in PC([t_0, \infty), R^n)$.

Furthermore we assume that f (t,0)=0, $I_k(t_k, 0) = 0$, k $\in Z_+$;then x$(t) \equiv 0$ is a solution of (1),which is called the zero solution. Moreover, we will only consider the solution $x(t, \sigma, \psi)$ of the system (1) which can be continued to ∞ from the right of σ.

Definition 1: The function $V : [\alpha, \infty) \times C \rightarrow R_+$ is said to belong to the class ν_0 if we have the following.

1)V is continuous in each of the sets $[t_{k-1}, t_k) \times C$,and $lim_{(t,w)\rightarrow(t_k^-,x)} V(t, w) = V(t_k^-, x)$ exists.

2) $V(t, x)$ is locally Lipschitzian in x and $V(t, 0) \equiv 0$.

Definition 2: Given a function $V : R^+ \times R^n \rightarrow R^+$, the upper right-hand derivative of V with respect to system (i) is defined by $D^+V(t, x(t))$ = lim $sup_{\delta\rightarrow 0^+} \frac{1}{\delta}[V(t+\delta, x(t+\delta)) - V(t, x(t))]$.

*Definition 3:*A zero solution of system(1) is exponentially stable if, for any initial data $x_{t_0} = \psi$,there exist constants $\mu > 0$,M≥ 0 such that

$$|x(t, t_0, \psi)| \leq M|\psi|e^{-\mu(t-t_0)}, t \geq t_0$$

III. MAIN RESULTS

In this section, we shall present and prove our main results. Our results complement and improve some of the existing results in literature.

Theorem 1: The trivial solution of (1) is exponentially stable if there exist functions $w_1, w_2 \in K, c \in C(R_+, R_+), p \in PC(R_+, R_+), V(t,x) \in v_0$ and some constants q>1,$\lambda > 0, \beta_k \geq 0, k \in Z_+$ s.t.

(i)$w_1(|x|) \leq V(t,x) \leq w_2(|x|), (t,x) \in [t_0+\alpha, \infty)\times R^n$

(ii)for all $(t_k, \varphi) \in R_+ \times$PC([α ,0], R^n), $V(t_k, x(t_k^-) + I_k(x(t_k^-))) \leq (1+\beta_k)V(t_k^-, x(t_k^-))$, with $\sum_{k=1}^{\infty} \beta_k < \infty$.

(iii)for any $\sigma \geq t_0$ and $\varphi \in$ PC([α ,0], R^n),if $V(t,\varphi(0)) \geq V(t+s, \varphi(s))e^{-\int_{t+\alpha}^{t} m(s)ds}$ for s$\in$[α ,0], $t \neq t_k$ and m(t) $\in$PC($[t_0+\alpha, \infty) \times R_+$) and $inf_{t \geq t_0+\alpha} m(t) \geq \lambda$ then $D^+V(t, \varphi(0) \leq -m(t)V(t, \varphi(0))$.

Proof: Let $x(t) = x(t, t_0, \psi)$ be the solution of the system (1) and V(t)=V(t,x(t)).

We shall show that V(t)$\leq w_2 \prod_{i=0}^{k-1}(1+\beta_i)\,|\psi| e^{-\int_{t_0}^{t} m(s)ds}, t \in (t_k, t_{k-1}), k \in Z_+$

where $\beta_0 = 0$.Let

$$Q(t)=\begin{cases} V(t)- w_2 \prod_{i=0}^{k-1}(1+\beta_i)\,|\psi| e^{-\int_{t_0}^{t} m(s)ds}, & t \in (t_k, t_{k-1}), k \in Z_+ \\ V(t)- w_2|\psi| e^{-\int_{t_0}^{t} m(s)ds}, & t \in (t_0+\alpha, t_0). \end{cases}$$

We need to show that Q(t)≤ 0 , $\forall t \leq 0$.It is clear that Q(t)≤ 0 for $t \in [t+\alpha, t_0)$. Since Q(t)$\leq v$(t)- $w_2|\psi| \leq 0$ by condition (i).

Take k=1.We shall show that Q(t) ≤ 0 for all $t \in [t_0, t_1)$.In order to do this we can let $\gamma > 0$ be arbitrary and show that Q(t)$\leq \gamma$ for $t \in [t_0, t_1)$.

Suppose not, then there exist some $t \in [t_0, t_1)$ so that Q(t)>γ.Let $t^* = \inf\{t \in [t_0, t_1) : Q(t) > \gamma\}$. Since Q(t) $\leq 0 < \gamma$ for $t \in [t_0+\alpha, t_0]$,we know that $t^* \in (t_0, t_1)$ .Note that Q(t) is continuous on $[t_0, t_1)$,then Q(t^*) $= \gamma$ and Q(t)$\leq \gamma$ for t$\in [t+\alpha, t^*]$.

Notice $V(t^*) = Q(t^*) + w_2|\psi| e^{-\int_{t_0}^{t^*} m(s)ds}$ and for s$\in [\alpha, 0]$,we have

$$V(t^*+s) = Q(t^*+s) + w_2|\psi| e^{-\int_{t_0}^{t^*+s} m(s)ds}$$

$$\leq \gamma + w_2|\psi|e^{-\int_{t_0}^{t^*+\alpha} m(s)ds}$$

$$\leq (\gamma + w_2|\psi|e^{-\int_{t_0}^{t^*} m(s)ds})\ e^{-\int_{t^*}^{t^*+\alpha} m(s)ds}$$

$$=\mathrm{V}(t^*)e^{\int_{t^*+\alpha}^{t^*} m(s)ds}$$

So by condition (iii),we have $D^+V(t^*) \leq -m(t^*)V(t^*)$,then we have

$$D^+Q(t^*) = D^+V(t^*) + m(t^*)w_2|\psi|e^{-\int_{t_0}^{t^*} m(s)ds}$$

$$\leq m(t^*)(V(t^*) - w_2|\psi|e^{-\int_{t_0}^{t^*} m(s)ds})$$

$$=-m(t^*)\gamma < 0$$

Which contradicts the definition of t^*,so we get Q(t) $\leq \gamma$ for all t$\in [t_0, t_1)$.Let $\gamma \to 0^+$.we have Q(t) $\leq$ 0 for t$\in [t_0, t_1)$.

Now assume that Q(t) $\leq$ 0 for t$\in[t_0, t_m), m \geq 1$.We shall show that Q(t) $\leq$ 0 for t$\in [t_0, t_{m+1})$.

By condition (ii) we have

$$\mathrm{Q}(t_m) = V(t_m) - w_2 \prod_{i=0}^{m}(1+\beta_i)\,|\psi|e^{-\int_{t_0}^{t_m} m(s)ds}$$

$$\leq (1+\beta_m)V(t_m^-) - w_2 \prod_{i=0}^{m}(1+\beta_i)\,|\psi|e^{-\int_{t_0}^{t_m} m(s)ds}$$

$$=(1+\beta_m)Q(t_m^-) \leq 0.$$

Let $\gamma > 0$ be arbitrary, we need to show that Q(t) $\leq \gamma$ for t$\in (t_m, t_{m+1})$. Suppose not,let $t^* = \inf\{t \in [t_m, t_{m+1}): Q(t) > \gamma\}$. Since Q($t_m) \leq 0 < \gamma$,by the continuity of Q(t),we get $t^* > t_m, Q(t^*) = \gamma$ and Q(t) $\leq \gamma$ for t$\in [t_0, t^*]$.

Since V$(t^*) = Q(t^*) + w_2 \prod_{i=0}^{m}(1+\beta_i)\,|\psi|e^{-\int_{t_0}^{t^*} m(s)ds}$,then for any s$\in [\alpha, 0]$,we have

$$\mathrm{V}(t^*+s) \leq Q(t^*) + w_2 \prod_{i=0}^{m}(1+\beta_i)\,|\psi|e^{-\int_{t_0}^{t^*+s} m(s)ds}$$

$$\leq \gamma + w_2 \prod_{i=0}^{m}(1+\beta_i)\,|\psi|e^{-\int_{t_0}^{t^*+\alpha} m(s)ds}$$

$$\leq (\ \gamma + w_2 \prod_{i=0}^{m}(1+\beta_i)\,|\psi|e^{-\int_{t_0}^{t^*} m(s)ds})e^{-\int_{t^*}^{t^*+\alpha} m(s)ds}$$

$$=\mathrm{V}(t^*)e^{\int_{t^*+\alpha}^{t^*} m(s)ds}$$

Thus by condition (iii),we have $D^+V(t^*) \leq -m(t^*)V(t^*)$,and then we have

$$D^+Q(t^*) = D^+V(t^*) + m(t^*)w_2 \prod_{i=0}^{m}(1+\beta_i)\,|\psi|e^{-\int_{t_0}^{t^*} m(s)ds}$$

$$\leq -m(t^*)(V(t^*) - w_2 \prod_{i=0}^{m}(1+\beta_i)\,|\psi|e^{-\int_{t_0}^{t^*} m(s)ds})$$

$$\leq -m(t^*)\gamma < 0$$

Again this contradicts the definition of t^*,which implies Q(t) $\leq \gamma$ for all t$\in [t_m, t_{m+1})$. Let $\gamma \to 0^+$.we have Q(t) ≤ 0 for all t$\in [t_m, t_{m+1})$.Thus by the method of induction,we get

V(t) $\leq w_2 \prod_{i=0}^{k-1}(1+\beta_i)\,|\psi| e^{-\int_{t_0}^{t} m(s)ds}$,t$\in [t_{k-1}, t_k), k \in Z_+$.

By condition (i)-(iii),we have

$$w_1|x| \leq V(t) \leq w_2 \prod_{i=0}^{k-1}(1+\beta_i)\,|\psi| e^{-\int_{t_0}^{t} m(s)ds} \leq w_2 M|\psi| e^{-\lambda(t-t_0)}, t \geq t_0,$$

which yields

$$|x| \leq (\frac{w_2 M}{w_1})|\psi|\, e^{-\lambda(t-t_0)},\ t \geq t_0$$

Where M=$\prod_{i=0}^{\infty}(1+\beta_i) < \infty$,Since $\sum_{k=1}^{\infty}\beta_k < \infty$.Thus the proof is complete.

Theorem 2: Assume that hypotheses(i)-(iv) are satisfied and there exist a function V$\in v_0$ and constants $\delta > 1, w_1 > 0, w_2 > 0$ and $\tau \leq \frac{\ln \delta}{\alpha}$ such that:

(i)$w_1(|x|) \leq V(t,x) \leq w_2(|x|)$

(ii)For all $t \neq t_k\ in\ R_+$whenever $\delta V(t,\varphi(0) \geq V(t+s,\varphi(s)$ for $s \in [\alpha, 0]$, $D^+V(t,\varphi(0) \leq -\tau V(t,\varphi(0))$

(iii) $(V(t_k,\varphi(0) + I_k(t_k,\varphi) \leq \omega_k\left(V\left(t_k^-,\varphi(0)\right)\right) where\ \varphi(0^-) = \varphi(0)$, and $\varphi_k(s)$, is continuous $0 \leq \omega_k(rs) \leq r\omega_k(s)$ holds for any r ≥ 0 and s ≥ 0 ,and there exist $\Upsilon \geq 1$ such that $\omega_k(\frac{\omega_{k-1}(\ldots(\omega_1(s)\ldots))}{s} \leq \Upsilon$. s>0, $k \in Z_+$.

Then the trivial solution of system (1) is exponentially stable.

Proof. By the same process as in the proof of Theorem 1.we can prove that zero solution of system (1) is exponentially stable.

IV. REFERENCES

[1] Xiaodi Li(2012), Further analysis on uniform stability of impulsive infinite delay differential equations,Applied Mathematics Letters 25 ,pp.133-137.

[2] Xilin Fu, Xiaodi Li(2009), Razumikhin-type theorems on exponential stability of impulsive infinite delay differential systems, Journal of Computational and Applied Mathematics 224,pp. 1–10

[3] Zhiguo Luo,Jianhua Shen(2001) , Stability and boundedness for impulsive functional differential equations with infinite delays ,Nonlinear Analysis 46,pp.475-493.

[4] G. Ballinger, X. Liu (1999), Existence and uniqueness results for impulsive delay differential equations, Dynam.Contin. Discrete Impuls. Systems 5,pp. 579–591.

[5] I.M. Stamova, G.T. Stamov(2001), Lyapunov–Razumikhin method for impulsive functional equations and applications to the population dynamics, J. Comput. Appl. Math. 130,pp. 163–171.

[6] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov(1989), Theory of Impulsive Differential Equations, World Scientific,Singapore.

[7] V.B. Kolmanovskii, V.R. Nosov(1986), Stability of Functional Differential Equations, Academic Press, London.

[8] S.K. Srivastava , Kanwalpreet Kaur(2013) , Stability of Impulsive Differential Equation with any Time Delay, International Journal of Innovation and Applied Studies, pp. 280-286

