Decomposition of Triangular Toeplitz Matrices Leading to Recurrence Relation

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Abstract: A non-unit bidiagonal matrix and its inverse with simple structures are introduced. These matrices can be constructed easily using the entries of a given non-zero vector without any computations among the entries. The matrix transforms the given vector to a column of the identity matrix. The given vector can be computed back without any round off error using the inverse matrix. By applying such matrices, a simple and direct factorization of a given non-singular triangular Toeplitz matrix is presented here. This factorization contributes to inversion of the triangular Toeplitz matrix in a convenient way. Another significant outcome of the factorization is that it establishes a recurrence relation among finite dimensional non-singular $k \times k$ triangular Toeplitz matrices, $k = 1, 2, \ldots, n$. The result can be easily extended to $n \times n$ symmetric triangular matrices.

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I. INTRODUCTION

Bjorck and Pereyra in 1970 used in their classical work [1] unit bidiagonal matrices with constant off-diagonal entries and diagonal matrices for the ***LU*** representation of the inverse of Vandermonde matrices. A non-unit bidiagonal matrix with row-wise constant entries having opposite signs is also used for representing the factors. A recent extension of this approach is adapted to Vandermonde like matrices in Nicholas Higham's book [2]. Regarding the stability of confluent Vandermonde systems, weak stability and weakly stable algorithm concepts are presented in [2]. Weakly stable algorithms solve the dual of non-confluent or confluent Vandermonde or Vandermonde like systems with good accuracy in floating point arithmetic, when there will be not much subtractive cancellations in the inverse Vandermonde ***UL*** representation. The desirable criterion for making a minimal amount of subtractive cancellation is that those individual factors of ***U*** and ***L*** have alternating sign pattern for $A=(a_{ij}); -1^{(i+j)}a_{ij} \geq 0$. The lower triangular components of ***L*** are bidiagonal matrices with row-wise constant entries and alternate sign patterns. Higham reports that these components will maintain alternating sign pattern if the points are distinct and arranged in increasing order. Note that Higham does not consider the properties of these matrices outside this stability and accuracy domain and later extended their use for deriving stable factors for Vandermonde like matrices [2] by extending the Bjorck and Pereyra factorization of Vandermonde matrices. It can also be noted that in these two works, the linear transformation that maps a given vector to a column of the identity matrix is not at all considered and inverse of this transformation is not utilized for the factorization of Vandermonde System matrices and other matrical properties are not taken into consideration. From a totally different background and new perception we are going to introduce the lower bidiagonal version of these matrices and present several interesting features with such matrices. An interesting quoting from Gasca and Pena [3] is as follows. “ At our knowledge, the uniqueness of different factorizations, which is a consequence of the uniqueness of elimination process, is a novelty in this type of results”. It is applicable in this context of introducing the proposed bidiagonal matrices for factorization of non-singular ***n X n*** triangular Toeplitz matrices.

Toeplitz systems arise in vital application areas such as digital signal processing, linear speech prediction, communication network queue etc. [8,9]. Toeplitz matrices have constant entries along the diagonals. They are subset of the class of persymmetric matrices. Persymmetric matrices are symmetric about their northeast-southwest diagonals. This discussion is centered on the theme that a non-singular triangular Toeplitz matrix is closely associated with the bidiagonal operator matrix and its inverse presented here. The reason is that all these matrices can be constructed from a given set of numbers. For example, if a Toeplitz matrix is also symmetric, then it can be defined by a given set of n real quantities, say $r_1, r_2,.., r_n$ as below

$$T_n(r_1,r_2,...,r_n)=\begin{bmatrix} r_1 & r_2 & \cdots & \cdots & r_{n-1} & r_n \\ r_2 & r_1 & r_2 & \cdots & \cdots & r_{n-1} \\ \vdots & r_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{n-1} & \vdots & \ddots & \ddots & \ddots & r_2 \\ r_n & r_{n-1} & \cdots & \cdots & r_2 & r_1 \end{bmatrix}$$

Typically T_4 can be presented as below.

$$T_4(r_1,r_2,r_3,r_4)=\begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ r_2 & r_1 & r_2 & r_3 \\ r_3 & r_2 & r_1 & r_2 \\ r_4 & r_3 & r_2 & r_1 \end{bmatrix}$$

The organization of the paper is as follows. First we will introduce the bidiagonal matrix, its inverse and basic features which make it an ideal choice for factorization of matrices**.** After that we will discuss the factorization of a given ***nXn*** non-singular triangular Toeplitz Matrix, representation of its factors in a convenient way using the bidiagonal matrix and its inverse, and inversion of the given triangular Toeplitz matrix. This is followed by computational cost of the approach and concluding remarks.

II. THE BIDIAGONAL MATRIX AND ITS FEATURES

Let a non-zero vector $x=[x_1 \quad x_2 \quad \cdots \quad x_n]^T; x_i \neq 0, i=1,2,..n$ be given. Consider the lower bidiagonal matrix and its inverse defined as below.

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{x}) = [\alpha_{ij}]; \alpha_{ij} &= 1/x_i \text{ ; for } \boldsymbol{i} = \boldsymbol{j} \text{ ; } \boldsymbol{i}, \boldsymbol{j} = 1,2,\ldots, \boldsymbol{n}. \\ \alpha_{ij} &= -1/x_i \text{ ; for } \boldsymbol{i} = \boldsymbol{j} + 1. \\ \alpha_{ij} &= 0 \text{ ; } \boldsymbol{i} > \boldsymbol{j} + 1 \text{ and } \boldsymbol{i} < \boldsymbol{j} + 1. \end{aligned} \tag{2.1}$$

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{x})^{-1} = [\beta_{ij}]; \beta_{ij} &= x_i \text{ ; for } \boldsymbol{i} \geq \boldsymbol{j} \text{ ; } \boldsymbol{i}, \boldsymbol{j} = 1,2,\ldots, \boldsymbol{n}. \\ \beta_{ij} &= 0 \text{ ; for } \boldsymbol{i} < \boldsymbol{j} + 1. \end{aligned} \tag{2.2}$$

Typical examples for the case $n=3$ is as below.

$$\boldsymbol{B}(\boldsymbol{x}) = \begin{bmatrix} 1/x_1 & & \\ -1/x_1 & 1/x_2 & \\ & -1/x_2 & 1/x_3 \end{bmatrix} \qquad \boldsymbol{B}(\boldsymbol{x})^{-1} = \begin{bmatrix} x_1 & & \\ x_2 & x_2 & \\ x_3 & x_3 & x_3 \end{bmatrix}$$

If we look at the columns in (2.2), these are the given vector itself and its projection to the subspaces of dimension $\boldsymbol{k}=\boldsymbol{n}-1$, $\boldsymbol{n}-2,\ldots,1$. These columns constitute a basis and hence can represent any given vector in a unique way. Since the first column itself is the very same vector, the linear combination can be only the entries from $\boldsymbol{e}_1$. This forms the elementary theory behind the factorization. Clearly $\boldsymbol{B}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{e}_1$ and $\boldsymbol{B}(\boldsymbol{x})^{-1}\boldsymbol{e}_1 = \boldsymbol{x}$. If in the given vector $\boldsymbol{x}$, $x_k \text{ ; } \boldsymbol{k} = 1,2,\cdots,\boldsymbol{j}$ are zeros and $x_k \neq 0 \text{ ; } \boldsymbol{k} = \boldsymbol{j}+1,\cdots,\boldsymbol{n}$ then the first $\boldsymbol{j}$ rows in $\boldsymbol{B}(\boldsymbol{x})$ can be set identical to that of the identity matrix and then $\boldsymbol{B}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{e}_{j+1}$ and $\boldsymbol{B}(\boldsymbol{x})^{-1}\boldsymbol{e}_{j+1} = \boldsymbol{x}$. In general $\boldsymbol{B}(\boldsymbol{x})$ has to be appropriately tuned with the rows and columns of the identity matrix so that mapping of $\boldsymbol{x}$ to a column of the identity matrix is possible. In any case, the mapping will be to another vector $\boldsymbol{y}$ whose entries will be consisting of **±1** and zeros. Accept that as discussed in J.H. Wilkinson [4], a negligibly small error, say $|\boldsymbol{e}| \approx 2^{-t}$, where the computer has $\boldsymbol{t}$ digits mantissa, is bound to occur. Still the mapping and inverse mapping will be always without any round off errors because of the structure of the matrix (2.2) and the presence of unity. See Plamen Koev [5] that relative accuracy will be affected when floating point subtractions are involved as cancellation of significant digits during subtraction of intermediate quantities. This is applicable to the proposed factorization also. But the intermediate quantity is exactly maintained in the inverse of the operator matrix. This structural feature can thus contribute to preserve the relative accuracy. Recall the remarks of Higham [2] about the association of these matrices with his definition of weak stability in maintaining accuracy and stability. These operator matrices can be tuned appropriately with the columns of the identity matrix in presence of zero elements of a column instead of row or column exchanges.

The matrices in (2.1) and (2.2) are the results of applying a sequence of column or row operations in corresponding diagonal matrices and these can be illustrated as follows.

Consider a lower triangular matrix

$$\begin{aligned} \boldsymbol{T}(1) = \boldsymbol{L}(\boldsymbol{l}_{ij}) \text{ ; } \boldsymbol{l}_{ij} &= 1 \text{ for } \boldsymbol{i} \geq \boldsymbol{j} \text{ ; } \boldsymbol{i}, \boldsymbol{j} = 1,2,\ldots, \boldsymbol{n}. \\ \boldsymbol{l}_{ij} &= 0 \text{ for } \boldsymbol{i} < \boldsymbol{j} + 1. \end{aligned} \tag{2.3}$$

Then, we have

$$\boldsymbol{T}(1)^{-1} = \boldsymbol{bidiag}(-1,1) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & \\ & \ddots & \ddots & \ddots & \\ \cdots & & 0 & -1 & 1 \end{bmatrix} \quad (2.4)$$

A. ***Proposition 2.1***

The matrices

$$\boldsymbol{B}(x)^{-1} = \boldsymbol{D}(x)\boldsymbol{T}(1)\,;\, \boldsymbol{B}(x) = \boldsymbol{T}(1)^{-1}\boldsymbol{D}(x)^{-1} \text{ where } \boldsymbol{D}(x) = \boldsymbol{diag}(x_1, x_2, \cdots, x_n) \quad (2.5)$$

It is evident that the matrices in (2.1) and (2.2) are derived from the corresponding diagonal matrices by elementary column operations and whenever a diagonal element is zero it is equivalent to the cancellation of the column operations with the particular diagonal element. Thus the column is reverted to the corresponding column of the identity matrix in (2.3) and (2.4).

B. ***Proposition 2.2***

From proposition (2.1) and from (2.3) and (2.4) it follows that

$$\boldsymbol{B}(x) = \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ -1 & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & -1 & 1 \end{bmatrix} \boldsymbol{diag}(1/x_1, \cdots 1/x_n) \quad (2.6)$$

$$\boldsymbol{B}(x)^{-1} = \boldsymbol{diag}(x_1, \cdots, x_n) \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ \cdots & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & 1 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & 1 \end{bmatrix} \quad (2.7)$$

Equations (2.6) and (2.7) are the elementary bidiagonal decomposition (**EBD**) of the matrices used in the factorization technique.

Consider the matrix

$$\boldsymbol{L}(x) = \boldsymbol{B}(x)^{-1}\boldsymbol{diag}(x)\boldsymbol{B}(x) \quad (2.8)$$

This is an interesting lower triangular matrix and this construction (2.8) is possible only when the entries of $\boldsymbol{x}$ are distinct and non-zero. A typical 4×4 matrix of (2.8) is as below.

$$\boldsymbol{L}(x) = \begin{bmatrix} x_1 & & & \\ (x_1-x_2)x_2/x_1 & x_2 & & \\ (x_1-x_2)x_3/x_1 & (x_2-x_3)x_3/x_2 & x_3 & \\ (x_1-x_2)x_4/x_1 & (x_2-x_3)x_4/x_2 & (x_3-x_4)x_4/x_3 & x_4 \end{bmatrix} \quad (2.9)$$

The matrix in (2.9) has an eigen system where in the general case eigen vector corresponding to x_1 is $\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, eigen vector corresponding to x_2 is $\begin{bmatrix} 0 & x_2 & \cdots & x_n \end{bmatrix}^T$ and so on and that corresponding to x_n is $\begin{bmatrix} 0 & 0 & \cdots & x_n \end{bmatrix}^T$. The diagonal entries will constitute the terms $(x_j - x_{j+1})$ of each column of the matrix (2.9) and the fractional terms will be determined by the entries of its eigen vectors. For example, $(x_j^n - x_{j+1}^n)$ will be the terms corresponding to its n^{th} power whereas the fractional terms will not be changing. Thus merely by looking at the matrix, one can easily derive the eigen system. The attraction is that its inverse and any power can be easily arrived at without any computations. In the open interval $(0,1)$, this system attains the minimum and maximum when the off-diagonal entries are uniformly approaching zero. This matrix corresponds to all strictly monotonic decreasing and increasing sequences in the interval $(0,1)$ and correspondence among such sequence matrices are realized by similarity transformation using appropriate diagonal matrices. Recall the remarks by Higham [2] that entries of the inverse Vandermonde lower triangular components should be distinct and in ascending order. This is a pointer to the association of the eigen vectors of matrices (2.9) with such special matrices which are basically generated out of given n distinct quantities.

C. *Proposition 2.3*

*Given a non-zero **n**-vector* $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T; x_i \neq 0, i = 1,2,..n$ *then* 2^{n-1} *bidiagonal matrices can be constructed with absolute values of the entries same as that of type (2.1), all of which will map **x** to* e_1.

Proof: Let B be a lower bidiagonal matrix and consider the equation

$$\alpha_1 x_k + \alpha_2 x_{k+1} = 0 \qquad (2.10)$$

In (2.10) let α_1 and α_2 be two adjacent entries of a row of B. Assume that α_2 is a diagonal element and α_1 is the corresponding sub-diagonal element in B. For the first row in B, there is only one unique choice as $\alpha_1 = 0; \alpha_2 = 1/x_1$. For the rest of the rows, assigning one of these unknowns a value, the other can be obtained. So for the remaining $2(n-1)$ entries, there are infinitely many choices. Here the choice with respect to the diagonal element is $\alpha_2 = 1/x_k; k = 1,2,..., n.$ Then the off diagonal elements will be obviously $\alpha_1 = -1/x_{k-1}; k = 2,3,..., n.$ Accordingly with this choice we have settled for the matrix (2.1). But $\alpha_1 = 1/x_{k-1}; \alpha_2 = -1/x_k$ also will satisfy equation (2.10). Hence with respect to each of the n-1 rows, the entries can be filled in 2 ways and thus the result follows.

One has the freedom to select a bidiagonal matrix of choice. Then there is a chance that resulting reduction process will be handicapped with the problem of inverting the bidiagonal matrix at every step in addition to disturbing the structural property of the given matrix. For example, P.V Sankar and A. K. Sen [7] report that the factorization algorithm proposed by them has the problem of inverting triangular matrices at every step. In the case of the proposed scheme, the operator matrix and its inverse can be easily constructed. These constructions do not call for any additional computations among the entries as in Neville or Gauss. With respect to the number of iterative steps to eliminate column elements, the proposed matrix completes it simultaneously in a single step as against the element-by-element reduction in Neville decomposition. Obviously the operator transforms the given vector to a column of the most stable identity matrix and in a stable way. In short, from the infinite set of bidiagonal matrices of (2.10), an ideal bidiagonal matrix for factorization of a given matrix is presented. The detailed results for factorizing a given matrix using these non-unit bi-diagonal matrices are presented in Nair [6].

III. FACTORIZATION OF TRIANGULAR TOEPLITZ MATRIX

Fu-Rong Lin,Wai-Ki Ching and M.K.Michael [10] present an approximate inversion method for triangular Toeplitz matrices based on trigonometric polynomial interpolation. To obtain an approximate inverse of high accuracy for a triangular Toeplitz matrix of size n, their algorithm requires two fast Fourier transforms (**FFT**) and one fast cosine transform of $2n$-vectors. Kenneth S. Berenhaut, Daniel C. Morton and Preston T. Fletcher [11] provide an improvement on norm bound for the inverse of a lower triangular Toeplitz matrix with nonnegative entries. Here a direct factorization process shall be introduced for inverting these matrices. The factorization will be restricted to the non-singular case so that $r_k\,;\ k=1,2,..,n$ are non-zero real quantities. The significant outcome of this factorization process is that it establishes a recurrence relation among finite dimensional nXn triangular Toeplitz matrices. Consider the $4X4$ triangular Toeplitz matrix, say, $R_4(r_1,r_2,r_3,r_4)$ where

$$R_4 = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ r_2 & r_1 & 0 & 0 \\ r_3 & r_2 & r_1 & 0 \\ r_4 & r_3 & r_2 & r_1 \end{bmatrix}$$

Using $B(x)^{-1}$ matrix, $R_n(r_1,r_2,\ldots.,r_n)$ can be factorized in an interesting way as presented below. Since $R_n(r_1,r_2,\ldots.,r_n)$ is completely defined by the n quantities $r_k\,;\ k=1,2,..,n$, for convenience, it may be factorized $R_4(r_1,r_2,r_3,r_4)$ and by analogy, generalize the result to nXn triangular Toeplitz matrices. So let it be proceeded with the factorization below.

STEP-1

$$R_4 = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ r_2 & r_1 & 0 & 0 \\ r_3 & r_2 & r_1 & 0 \\ r_4 & r_3 & r_2 & r_1 \end{bmatrix} = D_1 F_1 \qquad (3.1)$$

where $D_1 = diag(r_1,r_2,r_3,r_4)$

$$F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & r_1/r_2 & 0 & 0 \\ 1 & r_2/r_3 & r_1/r_3 & 0 \\ 1 & r_3/r_4 & r_2/r_4 & r_1/r_4 \end{bmatrix}$$

Now the factor F_1 can be further decomposed as $F_1 = M_1 K_1$ as given in (3.2) below.

$$F_1 = M_1 K_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/r_2 & 0 & 0 \\ 1 & 1/r_3 & 1/r_3 & 0 \\ 1 & 1/r_4 & 1/r_4 & 1/r_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r1 & 0 & 0 \\ 0 & r_2 - r_1 & r_1 & 0 \\ 0 & r_3 - r_2 & r_2 - r_1 & r_1 \end{bmatrix} \tag{3.2}$$

In the first factor of F_1 in (3.2), M_1, the lower right sub-matrix obtained by removing its first row and column is a ***3X3***, $B(x)^{-1}$ matrix where $x=[\ 1/r_2\ \ 1/r_3\ \ 1/r_4\ ]^T$. Consider the second factor K_1. Here the significance is that the lower right sub-matrix obtained similarly as in the case of M_1, is the triangular Toeplitz matrix, say, $R_3(r_1\ ,\ r_2 - r_1\ ,\ r_3 - r_2)$. The first factor M_1 can be easily factorized as below.

$$M_1 = N_1 T_3(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/r_2 & 0 & 0 \\ 1 & 0 & 1/r_3 & 0 \\ 1 & 0 & 0 & 1/r_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \tag{3.3}$$

Recall from Section-2, using proposition (2.1) the representation of $B(x)^{-1}$ matrix using its column equivalent matrix of the identity matrix. Here (3.3) is a similar representation of the matrix M_1. The inverse of the first factor, say N_1 of M_1 can be easily derived as given below.

$$N_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -r_2 & r_2 & 0 & 0 \\ -r_3 & 0 & r_3 & 0 \\ -r_4 & 0 & 0 & r_4 \end{bmatrix}$$

The net result is that started with $R_4\ (r_1,r_2,r_3,r_4)$, the above process factorized it as a product, consisting of another matrix K_1 as below and other matrices with known inverses.

$$K_1 = \left[\begin{array}{c|c} I_1 & 0 \\ \hline 0 & R_3(r_1, r_2 - r_1, r_3 - r_2) \end{array}\right]_3^1 \tag{3.4}$$

$$\qquad\quad \mathbf{1} \qquad\qquad\qquad \mathbf{3}$$

The whole of the step-1 can be thus summarized as

$$R_4(r_1,r_2,r_3,r_4) = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ r_2 & r_1 & 0 & 0 \\ r_3 & r_2 & r_1 & 0 \\ r_4 & r_3 & r_2 & r_1 \end{bmatrix} = D_1 N_1 T_3(1) K_1$$
$$= \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/r_2 & 0 & 0 \\ 1 & 0 & 1/r_3 & 0 \\ 1 & 0 & 0 & 1/r_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} K_1 \tag{3.5}$$

In (3.5.), the inverses of all the components are known except K_1. But this is not an issue as the recurrence relation is now clear. That is the steps above can be repeated with the triangular

Toeplitz component $R_3(r_1, r_2 - r_1, r_3 - r_2)$ of K_1 as below.

STEP-2

$$K_1 = D_2 F_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 - r_1 & 0 \\ 0 & 0 & 0 & r_3 - r_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & r_1/(r_2 - r_1) & 0 \\ 0 & 1 & (r2 - r1)/(r3 - r2) & r1/(r3 - r2) \end{bmatrix} \quad (3.6)$$

In the decomposition (3.6) of K_1 above, the matrix F_2 can be further factorized as in step-1 as given below.

$$F_2 = M_2 K_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1/(r_2 - r_1) & 0 \\ 0 & 1 & 1/(r_3 - r_2) & 1/(r3 - r_2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r_1 & 0 \\ 0 & 0 & r_2 - 2r_1 & r_1 \end{bmatrix} \quad (3.7)$$

The situation is similar to that in step-1. In M_2 of (3.7), the lower right sub-matrix obtained by removing the first two rows and columns is $B(x)^{-1}$ where $x=[r_2-r_1 \quad r_3-r_2]^T$. The lower right submatrix obtained in a similar way from K_2 is a triangular Toeplitz matrix, say $R_2(r_1, r_2-2r_1)$ respectively. As in Step-1, M_2 can be further factorized as $M_2 = N_2 T_2(1)$. Thus we have,

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1/(r_2 - r_1) & 0 \\ 0 & 1 & 0 & 1/(r3 - r_2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (3.8)$$

$$K_2 = \begin{bmatrix} I_2 & 0 \\ 0 & R_2(r_1, r_2 - 2r_1) \end{bmatrix}_{2 \quad\; 2}^{2 \quad\; 2} \quad (3.9)$$

The whole of step-2 can be thus summarized as

$$K_1 = D_2 N_2 T_2(1) K_2 \quad (3.10)$$

That is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & r_2-r_1 & r_1 & 0 \\ 0 & r_3-r_2 & r_2-r_1 & r_1 \end{bmatrix} \tag{3.11}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2-r_1 & 0 \\ 0 & 0 & 0 & r_3-r_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1/(r_2-r_1) & 0 \\ 0 & 1 & 0 & 1/(r_3-r_2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} K_2$$

Now, the third and final step-3 for factorizing K_2 is presented as follows.

STEP-3

The matrix K_2 can be easily decomposed as below

$$K_2 = D_{31} T_2(1) D_{32} \tag{3.12}$$

That is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r_1 & 0 \\ 0 & 0 & r_2-2r_1 & r_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r_1 & 0 \\ 0 & 0 & 0 & r_2-2r_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r_1/(r_2-2r_1) \end{bmatrix}$$

In (3.12) $T_2(1)$ is column equivalent to the identity matrix. D_{31} and D_{32} are diagonal matrices where D_{32} is identical with the identity matrix I except for the last diagonal element. Thus factorization of the *4X4* triangular Toeplitz matrix $R_4\ (r_1,r_2,r_3,r_4)$ is completed with the above step-3.

The result can be easily extended to $n \times n$ symmetric Toeplitz matrices because of Cholesky decomposition. Both the lower and upper triangular factors will be the same Cholesky Toeplitz triangular matrix in this case. Since the Toeplitz triangular matrix has this recurrence relation, the Corresponding Cholesky triangular matrix also has this recurrence relation. Hence it follows that a full $n \times n$ symmetric Toeplitz matrix has also this recurrence relation.

IV. SALIENT FEATURES AND COMPUTATIONAL COST

The salient features of the factorization of the triangular Toeplitz matrix proposed here are;

1) Each step j ends with a matrix K_j which has a triangular Toeplitz lower right sub-matrix component of dimension $n\text{-}j\ ;\ j=1,2,\ldots,n\text{-}1$. The matrix has its first j columns and rows identical with that of the identity matrix. That is this factorization exposes a recurrence relation between K_j and K_{j-1}.

2) At each step k, a diagonal matrix component $D_k\ ;\ k=1,2,\ldots,n\text{-}1$ is involved whose $k\text{-}1$ columns and rows are identical with the identity matrix.

3) The matrices $T_k(1)$ and N_k at step-k as presented in (3.3) or (3.8) will constitute a matrix M_k which has a lower right sub-matrix $B(x_k)^{-1};k=1,2,\ldots,n\text{-}2$.

4) The above sub-matrix makes it possible to decompose a component matrix as in feature-***i*** above. For example, it may be observed how $\boldsymbol{F_j}$ is decomposed as $\boldsymbol{M_j}$ and $\boldsymbol{K_j}$ in (3.2) or (3.7) for $\boldsymbol{j=1,2,\ldots,n}$.

5) At step-$\boldsymbol{(n-1)}$, the process of factorization converges as presented in (3.12). Of the three components of $\boldsymbol{K_{n-1}}$, it may be observed that two components $\boldsymbol{D_{n-1}}$ and $\boldsymbol{D_{n-1,1}}$ are diagonal matrices and $\boldsymbol{D_{n-1,2}}$ is identical with the identity matrix, except for the $\boldsymbol{n^{th}}$ diagonal element. The component $\boldsymbol{T_{n-1}(1)}$ is a column equivalent matrix of the identity matrix with all its elements as unity and its ***2X2*** lower right sub-matrix is $\boldsymbol{B(x)^{-1}}$ where $\boldsymbol{x=[1\ 1]^T}$.

In terms of computational cost, this factorization is just arranging a given triangular Toeplitz matrix as a product of factors typically illustrated in (3.5) which is also a recurrence relation. The factors consist of a diagonal matrix, a matrix with only diagonal elements and a column, a triangular matrix with all elements unity and another triangular matrix where the lower right block is a triangular Toeplitz matrix of dimension one less than the previous triangular Toeplitz matrix and its left upper block consists of the columns and rows of the identity matrix. Since this pattern is repeated, there is no computational procedures are involved in the factorization. When it comes to inverting the given matrix, the factors are to be inverted and multiplied. Since most of these factors are almost diagonal matrices and matrices with upper left block consisting of columns and rows of the identity matrix, it requires only $\boldsymbol{O(n^2)}$ flops.

V. CONCLUSION

It can be generalized the aspects discussed in the previous section about the triangular Toeplitz factorization by analogy to higher dimension. The activities are identical in the total of the $\boldsymbol{n-1}$ steps, except the $\boldsymbol{n-1^{th}}$ one. It is a recurrence relation existing among the triangular Toeplitz matrices, which is established by this factorization that made it possible to repeat the process where only the dimensional changes are to be taken care of. As a result the outcome $\boldsymbol{K_j}$ can be generated at $\boldsymbol{j^{th}}$ step from $\boldsymbol{K_{j-1}}$ by repeating similar activities for $\boldsymbol{j=1,2,\ldots,n-2}$. In short, the total framework is valid for a triangular Toeplitz matrix of dimension $\boldsymbol{n}$. This factorization can be thus applied to invert the given $\boldsymbol{n\ x\ n}$ triangular Toeplitz matrix in a direct and simple way. Because of the general Cholesky decomposition, the result can be easily extended to a given symmetric $\boldsymbol{n\ x\ n}$ Toeplitz matrix.

VI. REFERENCES

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