An Iterative-Algorithmically Modified use of Bernstein's Polynomial Operator for an Optimally-Close Polynomial Approximation using Computational Intelligence

Ashok Sahai

Department of Mathematics & Statistics, Faculty of Science & Technology, St. Augustine Campus
The University of the West Indies, Trinidad & Tobago; WEST INDIES
E-mails: sahai.ashok@gmail.com, shok.sahai@sta.uwi.edu

Abstract. In approximation theory the celebrated Weirstrass, K. (1885)'s theorem heralded an intermittent interest in polynomial approximation. Bernstein polynomial approximation operator was very popular for quite some time. Many modifications were tried by altering the weight-function therein, including some with probabilistic perspective it had. Some iterative algorithms have also been tried. In this paper one such algorithm, using the 'An Iterative Algorithmically Modified Use of Bernstein's Polynomial Operator for an Optimally-Close Polynomial Approximation Using Computational Intelligence' has been proposed and studied. The paper includes an 'Empirical Simulation Study' to bring forth the extent of 'Relative Gain in Efficiency' in approximation at each iteration, relative to the original and the proposed 'Iterative' 'Polynomial Approximation Operator Algorithm' using 'Bernstein Polynomial' developed & studied for some example-functions "$(5)^x$, exp (x), ln(2+x) & sin (2+x) " with a high 'Replication' of '55,555'. Maple 17 has been used in this simulation study.

Keywords: Polynomial Approximation, Bias, Mean-Squared-Error, Empirical Simulation Study & Replication

I. INTRODUCTION & PROPOSITIONS

Lot of problems in Science & Engineering could, essentially, be formulated as optimizing and approximation {[6], [8], [10], [12], [14], [16], [17] & [18]}. One would good to peruse [4] & [5], in this context. The celebrated Weirstrass, K. (1885)'s {[32]} theorem proved that any continuous function could be approximated by a suitable-degree polynomial, as closely as one is pleased with. Polynomial functions are, of course, extremely well-behaved. One of the proofs {[1], [2]} was based on the popular 'Bernstein Polynomial':

B_n (f; x) $=\sum_{k=0}^{k=n} w_k(x).f(k/n)$; $w_k(x) = \left({}^nC_k\right).x^k.(1-x)^{(n-k)}$; k=0 (1) n being the weight-function, WHERE 'f (.)' is a bounded continuous function in C [0, 1], and "f (k/n)", k=0 (1) n are known values of 'f(x)' at "n+1" equidistant 'Knots'. (1.1)

The 'Weirstrass, K. (1885)'s theorem' & its proof using ''Bernstein Polynomial' was seminal to a lot of active and intermittent interest in 'polynomial approximation' and in 'Bernstein Polynomial', in particular {[3], [7], [9], [11], [13], [15], [18], [19], [20], [21], [22], [23], [24], [26], [27], [28], [29], [30] & [31]}. This paper, also, concerns with the "Bernstein's Polynomial Approximation Operator". Incidentally, weight function 'w_k(x)' in the "Bernstein's Polynomial Approximation Operator", as in (1.1), has a 'Probabilistic' interpretation being the typical 'Binomial Distribution Term'. Thus, 'B_n (f; x)' is nothing but E (f (x)) ~ 'Mathematical Expectation' of 'f (x)' ~ 'x' following the 'Binomial Distribution' ~ "weighted Average of the n +1 known values with respective weights 'w_k(x)'.

Without any loss of generality, [a, b] ~ [0, 1] under suitable transformation of the study-variable. We divide [0, 1] into '*n'* equal intervals, using 'n+1'equi-distant 'knots'. Let $x_i = i/n$ for i = 0, 1... n. If the unknown function is called by 'f(x)', the 'Bernstein's Polynomial Approximation uses the values f (xi)'s [i = 1, 2… n] which are assumed to be known.

Now, we propose our "Iterative Algorithmically modified Bernstein's Polynomial Operators, using the 'Computational Intelligence'". Let us denote the 'Original/Usual Bernstein Polynomial' in (1.1) at our Iteration 'Zero'", i.e. say, B_n [0] (f; x) ≡ B_n (f; x) as "$UB_{n=2}$ [0] (f; x) & $UB_{n=3}$ [0] (f; x), respectively, for n = 2 & n = 3.

Our propositions of the "Iterative ['I' standing for the iteration # in the]-Algorithmically modified Bernstein's Polynomial, say $IMB_{n=2\omega n=3}$[I] (f; x) 'using $UB_{n=2}$ [0] (f; x) & $UB_{n=3}$ [0] (f; x)', and the 'Computational Intelligence'", are then simply as follows.

$UB_{n=2}$ [0] (f; x) & $UB_{n=3}$ [0] (f; x) are, respectively, 'B_n (f; x)' [As in (1.1)] for n = 2 & n = 3 (1.2)

$$IMB_{n=2\omega n=3}[1] (f; x) = (1+c)* UB_{n=3} [0] (f; x) – c* UB_{n=2} [0] (f; x) \quad (1.3)$$

$$IMB_{n=2\omega n=3}[2] (f; x) = (1+c)* IMB_{n=2\omega n=3}[1] (f; x) – c* UB_{n=3} [0] (f; x) \quad (1.4)$$

$$\& \ IMB_{n=2\omega n=3}[3] (f; x) = (1+c)* IMB_{n=2\omega n=3}[2] (f; x) – c* IMB_{n=2\omega n=3}[1] (f; x) \quad (1.5)$$

Herein, 'c' is the design-parameter of the proposed "Iteration-Algorithm", the optimal-value say 'c_0', is determined to be "0.813", using 'Computational Intelligence' per an extensive "Simulation-Study"!

II. THE EMPIRICAL SIMULATION STUDY

To illustrate the gain in efficiency of the '$IMB_{n=2\omega n=3}$[I] (f; x); I=1,2 &3'', and subsequently by using the operator $IMB_{n=2\omega n=3}$[2] (f; x), after each 'Iteration" of our proposed Iterative Algorithmic Improvement of Polynomial Approximation by our proposed ""Iterative Algorithmically modified Bernstein's Polynomial Operators, using the 'Computational Intelligence'", we have carried out an empirical study.

We have taken the *cases of n = 2, 3, 4 and 5 (i.e. n + 1 = 3, 4, 5 and 6 knots)* in the empirical study to numerically illustrate the relative gain in efficiency in using the Algorithm vis-`a-vis the Original 'MMSE B_n [0] (f; x)' & 'MMSE B_n [I] (f; x); I ~ Iteration # = 1, 2, …., for each example-case of the n-values.

Essentially, the empirical study is a simulation one in which inasmuch as we assume that the function to be approximated, namely *f* (*x*), is known to us. We have confined ourselves to illustrating relative gain in efficiency by Iterative Improvement for the following four functions:

$$f(x) = 5^x, \ exp(x), \ \ln(2 + x) \ \&, \ sin(2 + x)$$

To illustrate the potential improvement with our proposed Algorithm, with only THREE Iterations, the numerical values of the NINE quantities – Three Percentage Relative Errors (*PREs*) corresponding to Improvement Iteration (I ≡ 1, *or* 2, *or* 3) ~ *PRE*{$IMB_{n=2\omega n=3}$[I] (f; x)}, TWO for the Original 'Bernstein Operator' *PRE* {UB_n [0] (f; x)} for n = 2 & n = 3, and the FOUR corresponding Percentage Relative Gains (*PRGs*) in using our Iterative Algorithmic 'MMSE Bernstein Operators $IMB_{n=2\omega n=3}$[I] (f; x)' in place of the Original 'Bernstein Operator' $UB_{n=2}$ [0] (f; x), namely *PRG*{ $IMB_{n=2\omega n=3}$[I] (f; x)}; I = 1(1)3), and also by using '$UB_{n=3}$ [0] (f; x)', rather than '$UB_{n=2}$ [0] (f; x)' . These quantities are defined as follows. The *PRE* using (Original & Iterative) Bernstein (Polynomial) "•", namely "{$IMB_{n=2\omega n=3}$[I] (f; x)}" and $UB_{n=2}$ [0] (f; x) & $UB_{n=3}$ [0] (f; x)" using n intervals in [0, 1], i.e. [(k − 1)/n, k/n]; k = 1(1) n:

$$\textbf{\textit{PRE}}\ \{\bullet\} = \left\{ \frac{abs[\int_0^1 f(x)dx - \int_0^1 \{\bullet\}dx]}{\int_0^1 f(x)dx} \right\} \textbf{x100\%;} \qquad (2.1)$$

The *PRG* by using the Improvement Iteration (*I* # 1, *or* 2, *or* 3) $IMB_{n=2\omega n=3}$[I] (f; x) & by using $UB_{n=3}$ [0] (f; x) over using the "Original" Bernstein (Polynomial) $UB_{n=2}$ [0] (f; x), using n intervals in [0, 1], i.e.

$$\textbf{\textit{PRG}}\ \{\bullet\} = = \left\{ \frac{abs[\mathrm{PRE}\{\mathrm{UB}_{n=2}[0](f;x)\} - \mathrm{PRE}(\bullet)]}{\mathrm{PRE}\{\mathrm{UB}_{n=2}[0](f;x)\}} \right\} \textbf{x100\%};$$

By using [I ≡ 1 (1) 3] $IMB_{n=2\omega n=3}$[I] (f; x) & by using $UB_{n=3}$ [0] (f; x) over using the "Original" Bernstein (Polynomial) $UB_{n=2}$ [0] (f; x). (2.2)

III. CONCLUSION

Thus, NINE numerical quantities have been computed using *Maple Release 17*, for all the four illustrative functions (5^x, exp (x), ln (2 + x), sin (2 + x), and) mentioned in Section 2. These values are, respectively, tabulated in Table A. 1 [Appendix].

The PREs for our 'Iteratively Algorithmically Modified Bernstein Polynomial' Approximators are PROGRESSIVELY lower on each subsequent iteration, as compared to that for the Original Bernstein Polynomial Approximator, for all the illustrative functions. The PRGs due to the use of our proposed 'Iterative Algorithmic MMSE Bernstein Polynomial' Approximators rather than that of the Original Bernstein Polynomial Approximator are also PROGRESSIVELY increasing on each subsequent iteration, for all the illustrative functions.

Lastly, it is very heartening to note that *when we use (n = 3) intervals, i.e. 4 KNOTS* only for the polynomial approximation, the *PRG* becomes *almost 100%* for the third iteration, for all FOUR functions! Otherwise also, the speed of convergence is highly accelerated by the Iteratively Algorithmically improvement by proposed Modified Bernstein Polynomial using the 'Computational Intelligence'.

It could also be noted that this perspective of the Iterative Improvement could be applied to any Polynomial Approximator, other than Bernstein Polynomial; more particularly to those belonging to the class of Positive Linear Operators, as they admit to the Statistical perspective 'MMSE' rather more readily!

IV. REFERENCES

[1] Bernstein, S. N. (1912). D´emonstration du th´eor`eme de Weierstrass fond´ee sur le calcul des probabilities. Comm. Soc. Math. Kharkov, 13:1–2.

[2] Bernˇsteˇın, S. N. (1932). Compl´etement `a l'article de E. Voronowskaja. C. R. Acad. Sci. U.R.S.S., pages 86–92, 1932.

[3] Butzer, P. L. (1953). Linear combinations of Bernstein polynomials. Canadian J. Math., 5:559–567.

[4] Carothers, N.L. (1998). *A Short Course on Approximation Theory*. Bowling Green State University, Bowling Green, OH.

[5] Carothers, N.L. (2000). *Real Analysis*. Cambridge University Press.

[6] Cheney, W. & Kincaid, D. (1994). *Numerical Mathematics and Computing*. Brooks/Cole Publishing Company.

[7] Costabile, F., M. I. Gualtieri, and S. Serra (1996). Asymptotic expansion and extrapolation for Bernstein polynomials with applications. BIT, 36(4):676–687, 1996.

[8] Dunham Jackson (1930). The theory of approximation, volume 11. Amer. Math. Soc. Coll. Publ. 17.

[9] Frentiu, M. (1970). Linear combinations of Bernˇsteˇın polynomials and of Mirakjan operators. Studia Univ. Babe¸s-Bolyai Ser. Math.-Mech., 15(1):63–68, 1970.
[10] Hartley, P.J. & Wynn-Evans, A. (1997). *A Structured Introduction to Numerical Mathematics*. Stanley Thornes.
[11] Hedrick, E.R. (1927). The significance of Weirstrass theorem. *The Amer. Math. Month*, 20, 211-213.
[12] Lorentz, G.G. (1986). *Approximation of Functions*. Chelsea.
[13] Lorentz, G. G. (1986). Bernstein polynomials. Chelsea Publishing Co., New York, second edition.
[14] May, C. P. (1976). Saturation and inverse theorems for combinations of a class of exponential-type operators. Canad. J. Math., 28(6):1224–1250.
[15] Phillips, George M. (1997). On generalized Bernstein polynomials. In Approximation and optimization, Vol. I (Cluj-Napoca, 1996), pages 335–340. Transilvania, Cluj-Napoca.
[16] Polybon, B.F. (1992). Applied Numerical Analysis. PWS-KENT.
[17] Popoviciu, T. (1935). Sur l'approximation des fonctions convexes d'ordre sup´erieur. Mathematica (Cluj), 10:49–54.
[18] Richards, W. A., A Sahai & M. R. Acharya (2010). An efficient polynomial approximation to the normal distribution function and its inverse function. Journal of Mathematics Research 2 (4), 47.
[19] Sahai, A. (2011). An Iterative Reduced-Bias Algorithm for a Dual-Fusion variant of Bernstein's Operator. International Journal of Mathematical Archive (IJMA) ISSN 2229-5046 2 (3).
[20] Sahai, A. (2011). Efficient Quadrature Using Bernstein's Polynomial Weights via Fusion of Two Dual-Perspectives. International Journal of Latest Trends in Mathematics 1 (1).
[21] Sahai, A. & S. Verma (2009). Efficient quadrature operator using dual-perspectives-fusion probabilistic weights. International Journal of Engineering and Technology 1 (1), 1-8.
[22] Sahai, A., S. Wahid & A. Sinha (2006). A positive linear operator using probabilistic approach Journal of Applied Sciences 6, 2662-2665.
[23] Sahai, A. (2004). An iterative algorithm for improved approximation by Bernstein's operator using statistical perspective. Applied Mathematics and Computation, 149, 327-335.
[24] Sahai, A., R. P. Jaju &, P. M. Mashwama (2004). A new computerizable quadrature formula using probabilistic approach. Applied mathematics and computation 158 (1), 217-224.
[25] Searls, Donald T. (1964), "The Utilization of a known Coefficient of Variation in the Estimation Procedure", Journal of the American Statistical Association, 59, 1225-1226.
[26] Sheilds, A. (1987). Polynomial Approximation, The Math. Intell. 9 (3), 5–7.
[27] Sofiya Ostrovska. (2003). q-Bernstein polynomials and their iterates. J. Approx. Theory, 123(2):232–255.
[28] Sorin G. Gal. (2008). Shape-Preserving Approximation by Real and Complex Polynomials. Birkh¨auser, Boston, Basel, Berlin.
[29] Voronovskaya, E. (1932). D´etermination de la forme asymptotique d'approximation des fonctions par les polynˆomes de M. Bernstein,. Doklady Akademii Nauk SSSR, pages 79–85.
[30] Wahid, S. A., A Sahai & MR Acharya (2009). A Computerizable Iterative-Algorithmic Quadrature Operator Using an Efficient Two-Phase Modification of Bernstein Polynomial. International Journal of Engineering and Technology 1 (3), 104-108
[31] Wang, Heping and XueZhi Wu (2008). Saturation of convergence for q-Bernstein polynomials in the case $q \geq 1$. J. Math. Anal. Appl., 337(1):744–750.
[32] Weierstrass, K. (1885). Uber die analytische Darstellbarkeit sogenannter willkurlicher Functionen einer reellen Veranderlichen. Sitzungsberichteder Koniglich Preussischen Akademie der Wissenshcaften zu Berlin, (pp. 633–639 & 789–805)

APPENDIX:

Table A.1. Relative Efficiency/Gain (%) For Usual Bernstein &IMBernstein [I] Using "Computational Intelligence".

Items↓ *n* →	5^x	**exp (x)**	**ln (2+x)**	**sin (2+x)**
PRE {$UB_{n=2}$ [0] (f; x)}	10.46200038	4.115693005	0.754126955	4.219915264
PRE {$UB_{n=3}$ [0] (f; x)}	6.967203690	2.742579548	0.502544991	2.814637336
PRE {$IMB_{n=2\omega n=3}$[1] (f; x)}	4.125933988	1.626238231	0.298008854	1.672146392
PRE {$IMB_{n=2\omega n=3}$[2] (f; x)}	1.815981666	0.718652831	0.131720969	0.743301239
PRE {$IMB_{n=2\omega n=3}$[3] (f; x)}	0.062009510	0.019214136	0.003471075	0.011849876
PRG {$UB_{n=3}$[0] (f; x)}	33.40466988	33.36287365	33.36069113	33.30109351
PRG {$IMB_{n=2\omega n=3}$[1] (f; x)}	60.56266643	60.48689178	60.48293297	60.37488226
PRG {$IMB_{n=2\omega n=3}$[2] (f; x)}	82.64211814	82.53871632	82.53331645	82.38587288
PRG {$IMB_{n=2\omega n=3}$[3] (f; x)}	99.40728821	99.53314944	99.53972276	99.71919161

