Impulsive Integrated Pest Management Model with a Holling Type II Functional Response

Bhanu Gupta[1], Amit Sharma[2]

[1]P.G. Department of Mathematics, Jagdish Chandra DAV College, Hoshiarpur, India
[2]P.G. Department of Mathematics, Jagdish Chandra DAV College, Hoshiarpur, India
[2]IKG PTU, PhD Scholar, Kapurthala, India

Abstract: In this paper, a prey predator impulsive mathematical model of integrated pest management is established, in which infected prey and predator (natural enemy) are released impulsively. By using the Floquet's theory for impulsive differential equations, small-amplitude perturbation methods and comparison techniques, we investigate the local and global stability of the susceptible pest-eradication periodic solution.

Keywords: Susceptible prey; Infected prey; Predator; Impulse.

I. INTRODUCTION

Mathematical modeling is a technique in which various natural processes and phenomena are converted into mathematical terms (expressions) and then studied for their solutions, and continually refined over a period of time to get more and more accurate and efficient results. During last few years mathematical modeling has played an important role in studying the biological, pharmaceutical and agricultural processes.

From thousands of years, pest control in agriculture has been a major concern for farmers. People in different era used their own methodologies and technologies to control the pests which destroy the plants and ultimately reduce crop production. However with the passage of time, a number of pest control strategies are available to farmers such as physical control, biological control and chemical control and remote sensing.

In biological control technique, the pests are controlled via manipulating the nature in which some organisms (predators or infected pests) are released, which consume or infect the pest population and ultimately improve the crop production. In chemical control, the pesticides are used to control the pests and are very effective as these can kill the pests rapidly but the extensive use of these pesticides are creating major health problems. So the time demands the use of more and more biological control to control the pests destroying crops. A number of authors studied the management of pest control using biological technique [1, 2, 8, 9, and 10].

Many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. The impulsive systems of differential equation are an adequate apparatus for the mathematical modeling of numerous processes and phenomena studied in biology, economics and technology etc. As in pest control strategies discussed above (chemical and biological), the pesticide, the infected pests or predators are released impulsively, so the pest management can be very efficiently studied through modeling by impulsive differential equations.

The study of pest management using impulsive differential equations was started in 2005 due to Zhang et al [7]. They modeled the system as:

$$\begin{cases} x'(t) = x(t)\big(1 - x(t)\big) - \frac{a_1 x(t)\, y(t)}{1+b_1 x(t)} \\ y'(t) = \frac{a_1 x(t)\, y(t)}{1+b_1 x(t)} - \frac{a_2 y(t) z(t)}{1+b_2\, y(t)} - d_1 y(t) \\ z'(t) = \frac{a_2 y(t) z(t)}{1+b_2\, y(t)} - d_2 z(t) \end{cases} \quad t \neq nT$$

$$\begin{cases} x(t+) = x(t) \\ y(t+) = y(t) \\ z(t+) = z(t) + p \end{cases} \quad t = nT$$

where $x(t)$, $y(t)$ and z(t) are the population density of prey, predator and top predator at time t respectively and p is the amount of the predator that is introduced into the population at periodic intervals of length T.

The authors established the conditions for global asymptotic stability of pest extinction periodic solution $(1,0,\bar{z})$ as well as for the permanency of the system. They proved that periodic solution $\big(1,0,\tilde{z}(t)\big)$ is locally asymptotically stable when $T < \frac{a_2 p(1+b_1)}{d_2(a_1 - b_1 d_1 - d_1)}$. and system is globally asymptotically stable.

Shi and Chan [6] studied an impulsive prey predator model with disease in prey for purpose of integrated pest management. Here the authors derived a sufficient condition for the global stability of susceptible pest eradication periodic solution.

The aim of this paper is to study a Holling II functional responses prey predator impulsive mathematical model of integrated pest management in which both infected prey and predator (natural enemy) are released impulsively. The local stability of pest extinction periodic solution is studied using Floquet theory and global stability of pest extinction periodic solution is studied by using comparison principle of impulsive differential equations.

II. MATHEMATICAL MODEL

Assumptions:

A_1 : Due to disease in pest population the total pest population is divided into two classes, susceptible pest population and infected pest population.

A_2 : The incidence rate among susceptible and infected pest population is Holling type II i.e., $\frac{\alpha S(t)\, Z(t)}{1+xS(t)}$ where α is the contact number per unit time of infected pest with susceptible pest.

A_3 : The predator attacks susceptible pests only and the predation functional response is again Holling type II i.e., $\frac{\beta S(t) I(t)}{1+yS(t)}$ where β is the contact number per unit time of predator with susceptible pest.

A_4 : At time $t = nT, n \in Z_+ = \{1,2,3,,\ldots\}$ the infected pest and predator are released periodically with amount u and v respectively, $u > 0, v > 0$.

In this paper, we study the following model for integrated pest management

$$\begin{cases} S'(t) = S(t)\big(1 - S(t)\big) - \frac{\alpha S(t)Z(t)}{1+xS(t)} - \frac{\beta S(t)I(t)}{1+yS(t)}, \\ I'(t) = \frac{\beta S(t)I(t)}{1+yS(t)} - d_1 I(t), \\ Z'(t) = \frac{\alpha S(t)Z(t)}{1+xS(t)} - d_2 Z(t), \end{cases} \quad t \neq nT,$$
$$\begin{cases} \Delta S(t) = 0, \\ \Delta I(t) = u, \\ \Delta Z(t) = v. \end{cases} \quad t = nT, n \in Z_+ = \{1,2,3,,\dots\} \qquad (1)$$

Where

$S(t)$: Density of susceptible pest at time t.
$I(t)$: Density of infected pest at time t.
$Z(t)$: Density of predator (natural enemy) at time t.
d_1: Natural death rate of infected pest.
d_2: Natural death rate of predator (natural enemy).
α : The contact number per unit time of predator with susceptible pest.
β : The contact number per unit time of infected pest with susceptible pest.
u : The amount of infected pest released periodically.
v: The amount of predator released periodically.

$\Delta S(t) = S(t^+) - S(t)$,

$\Delta I(t) = I(t^+) - I(t)$,

$\Delta Z(t) = Z(t^+) - Z(t)$.

T is period of impulsive effect.

III. PRELIMINARIES

$R_+ = [0,\infty)$ and $R_+^3 = \{x = (x_1,x_2,x_3) \in R_+^3 : x_1,x_2,x_3 > 0\}$.
Let $V: R_+ \times R_+^3 \to R_+$. Then V is said to belong to class V_0 if

(i) V is continuous in $(nT,(n+1)T] \times R_+^3$ and for each $x \in R_+^3, n \in Z_+ = \{1,2,3,,\dots\}$ and the $\lim_{(t,y)\to(nT^+,x)} V(t,y) = V(nT^+, x)$ exists and is finite.

(ii) V is locally Lipschitzian in x.

Definition 3.1 For $V \in V_0$ and $(t,x) \in (nT,(n+1)T] \times R_+^3$, the upper right Dini derivative of $V(t,x)$ with respect to the impulsive differential system (1) is defined as
$D^+ V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}\big[V\big(t+h, x+hf(t,x)\big) - V(t,x)\big]$.

Definition 3.2 System (1) is said to be permanent if there exists a compact region $D \in \operatorname{int} R_+^3$ such that every solution of system (1) with positive initial values will eventually enter and remain in region D. The solution of system (1) denoted by $X(t) = \big(S(t), I(t), Z(t)\big): R_+ \to R_+^3$ is continuously differentiable on $(nT,(n+1)T] \times R_+^3, n \in Z_+ = \{1,2,3,,\dots\}$ and the limit $X(nT^+) = \lim_{t\to nT^+} X(t)$ exists and is finite for $n \in Z_+$. The global existence and uniqueness of solution of system (1) is guaranteed by the smoothness properties similar as in [5]. Below we state some lemmas whose proofs are obvious.

Lemma 3.1 Suppose that $X(t)$ is a solution of (1) with $X(0^+) \geq 0$ for all $t > 0$. Further, if $X(0^+) > 0$ then $X(t) > 0$ for all $t > 0$.

Lemma 3.2 [5] Let $V: R_+ \times R_+^3 \to R$ and $V \in V_0$. Assume that

$$\begin{cases} D^+V(t,x) \le g(t, V(t,x)), & t \ne nT, \\ V(t, X(t^+)) \le \psi_n V(t, X(t)), & t = nT, \end{cases}$$

where $g: R_+ \times R_+ \to R$ is continous in $(nT, (n+1)T] \times R_+$ and for each $v \in R_+^3, n \in Z_+$

$$\lim_{(t,y)\to(nT^+,v)} g(t,y) = g(nT^+, v)$$

exists and is finite. Let $\psi_n: R_+ \to R_+$ is non-decreasing and R(t) be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} u'(t) = g(t,u), \ t \ne nT, \\ u(t^+) = \psi_n(u(t)), t = nT, \\ u(0^+) = u_0, \end{cases}$$

defined on $[0, \infty)$. Then $V(0^+, x_0) \le u_0$ implies that $V(t, x(t)) \le R(t), t \ge 0$, where x(t) is any solution of system (1).

IV. BOUNDEDNESS

In this section, we prove the boundedness of the system (1).

Lemma 4.1 [5] Let the function $m \in PC'[R^+, R]$ and m(t) be left-continuous at $t_k, k = 1,2,3, \ldots\ldots\ldots$ satisfy the inequalities

$$\begin{cases} m'(t) \le p(t)m(t) + q(t), t \ge t_0, t \ne t_k, \\ m(t_k') \le d_k m(t_k) + b_k, t = t_k, k = 1,2,3, \ldots \end{cases} \tag{2}$$

where $p, q \in PC'[R^+, R]$ and $d_k \ge 0, b_k$ are constants, then

$$m(t) \le m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^t p(s)ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_0<t_j<t} d_j \exp\left(\int_{t_0}^t p(s)ds\right)\right) b_k + \int_{t_0}^t \prod_{s<t_k<t} d_k \exp\left(\int_s^t p(\sigma)d\sigma \ q(s)\, ds\right), \quad t \ge t_0. \tag{3}$$

If all the directions of inequalities in (2) are reversed, the inequality (3) also holds true for the reversed inequality.

Theorem 4.2 There exist a positive constant L such that $S(t) \le L, I(t) \le L, Z(t) \le L$, for each solution $(S(t), I(t), Z(t))$ of system (1) with positive initial values, where t is large enough.

Proof. Define a function V such that

$$V(t) = S(t) + I(t) + Z(t).$$

Then for $t \ne nT$,

$$D^+V(t) + dV(t) = S'(t) + I'(t) + Z'(t) + d(S(t) + I(t) + Z(t))$$

Let $d = \min\{d_1, d_2\}$.

We obtain $D^+V(t) + dV(t) \le (1+d)S(t) - S^2(t) \le M_0$, where $M_0 = \frac{(1+d)^2}{4}$.

When $t = nT, V(t^+) = V(t) + u + v$.

Using Lemma 4.1, we get

$$V(t) = V(0)e^{-dt} + \int_0^t M_0\, e^{-d(t-s)} ds + \sum_{0<kT<t} (u+v)e^{-d(t-kT)} \to \frac{M_0}{d} + \frac{(u+v)e^{dT}}{e^{dT}-1}, \text{as } t \to \infty.$$

Consequently, by the definition of V(t) we obtain that each solution of (1) with positive initial values is uniformaly ultimately bounded. This completes the proof.

V. STABILITY

In this section, we study the stability of pest eradication periodic solution of system (1) using Floquet theory of impulsive differential equations. The condition for global attraction of pest eradication period is also established.

Lemma 4.3. System

$$u'(t) = -w\,u(t), \quad t \neq nT,$$
$$\Delta u(t) = \mu, \quad t = nT. \qquad (4)$$

has a positive solution $u^*(t)$, for every solution $u(t)$ of this system with positive value $u(0^+)$, $|u(t) - u^*(t)| \to 0$ as $t \to \infty$, wher $u^*(t) = \frac{\mu e^{-w(t-nT)}}{1-e^{-wT}}$

$$u^*(0^+) = \frac{\mu}{1-e^{-wT}}.$$

Proof. The proof is obvious, in fact, since the solution of (4) is

$$u(t) = \left(u(0^+) - \frac{\mu}{1-e^{-wT}}\right) e^{-wT} + u^*(t). \quad nT < t \leq (n+1)T.$$

When $S(t) \equiv 0$ for all $t \geq 0$, we get the subsystem of system (1)

$$\begin{cases} I'(t) = -d_1 I(t), \\ Z'(t) = -d_2 Z(t), \end{cases} t \neq nT, \qquad \begin{cases} \Delta I(t) = u, \\ \Delta Z(t) = v. \end{cases} t = nT, n \in Z_+ = \{1,2,3,,\ldots\} \qquad (5)$$

In this system, we can see there is no relation between $I(t)$ and $Z(t)$. Thus, we can solve them independently. By Lemma 3.6, we get the following result.

Theorem 4.3 System (5) has a unique positive periodic solution

$$I^*(t) = \frac{u e^{-d_1(t-nT)}}{1-e^{-d_1T}},$$
$$Z^*(t) = \frac{v e^{-d_2(t-nT)}}{1-e^{-d_2T}}, \text{ for } nT < t \leq (n+1)T.$$

Where

$$I^*(0^+) = \frac{u}{1-e^{-d_1T}},$$
$$Z^*(0^+) = \frac{v}{1-e^{-d_2T}}.$$

In addition for every solution of the system (5) with initial values $I(0^+) > 0, Z(0^+) > 0$, it follows that $I(t) \to I^*(t), Z(t) \to Z^*(t)$, as $t \to \infty$.

Thus, the complete expression for the susceptible pest-eradication solution of system (1) is obtained as $(0, I^*(t), Z^*(t))$.

Theorem 5.1 Let $(S(t), I(t), Z(t))$ be any solution of (1) (i) The trivial solution $(0,0,0)$ is unstable. (ii) If $T < \left(\frac{\alpha v}{d_2} + \frac{\beta u}{d_1}\right)$, then the susceptible pest eradication periodic solution $(0, I^*(t), Z^*(t))$ is locally asymptotically stable.

Proof: (i) To prove the local stability of trivial solution $(0,0,0)$, we use small-amplitude perturbation method. Let

$$S(t) = p(t),$$
$$I(t) = q(t),$$
$$Z(t) = r(t),$$

where p(t), q(t), r(t) are small perturbations. Then system (1) can be linearized by using Taylor expansions after neglecting higher-order terms as:

$$\begin{cases} \left.\begin{aligned} p^{'}(t) &= p(t), \\ q^{'}(t) &= -d_1 I^*(t), \\ r^{'}(t) &= -d_2 Z^*(t), \end{aligned}\right\} & t \neq nT \\ \left.\begin{aligned} p(nT^+) &= p(nT), \\ q(nT^+) &= q(nT), \\ r(nT^+) &= r(nT), \end{aligned}\right\} & t = nT, n \in Z_+ = \{1,2,3,,\dots\} \end{cases} \quad (6)$$

Let $\emptyset(t)$ be the fundamental solution matrix of (6). Then $\emptyset(t)$ satisfy

$$\frac{d\emptyset(t)}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix} \emptyset(t),$$

$\emptyset(0) = I_3$ is the identity matrix. Hence the fundamental solution matrix is

$$\emptyset(t) = \begin{pmatrix} e^{T} & 0 & 0 \\ 0 & e^{-d_1 T} & 0 \\ 0 & 0 & e^{-d_2 T} \end{pmatrix}$$

Also, the fourth, fifth and sixth equations in (6) read as

$$\begin{pmatrix} p(nT^+) \\ q(nT^+) \\ r(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p(nT) \\ q(nT) \\ r(nT) \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \emptyset(T).$$

Therefore the eigen values of M are

$\lambda_1 = e^{-d_2 T} < 1, \lambda_2 = e^{-d_1 T} < 1, \quad \lambda_3 = e^{T} > 1.$

Since $\lambda_3 > 1$, trivial solution $(0,0,0)$ is unstable.

(ii) Now for local stability of periodic solution $(0, I^*(t), Z^*(t))$, the small-amplitude perturbation implies

$$S(t) = p(t),$$
$$I(t) = q(t) + I^*(t),$$
$$Z(t) = r(t) + Z^*(t),$$

where p(t), q(t), r(t) are small perturbations. Then system (1) can be linearized by using taylor expansions and after neglecting higher-order terms we get

$$\begin{cases} \left.\begin{aligned} p^{'}(t) &= p(t) - \alpha\, p(t) Z^*(t) - \beta p(t) I^*(t), \\ q^{'}(t) &= \beta p(t) I^*(t) - d_1 q(t) - d_1 I^*(t), \\ r^{'}(t) &= \alpha\, p(t) Z^*(t) - d_2 r(t) - d_2 Z^*(t), \end{aligned}\right\} & t \neq nT \\ \left.\begin{aligned} p(nT^+) &= p(nT), \\ q(nT^+) &= q(nT), \\ r(nT^+) &= r(nT), \end{aligned}\right\} & t = nT, n \in Z_+ = \{1,2,3,,\dots\} \end{cases} \quad (7)$$

Let $\emptyset(t)$ be the fundamental solution matrix of (7). Then $\emptyset(t)$ satisfy

$$\frac{d\emptyset(t)}{dt} = \begin{pmatrix} 1 - \alpha Z^*(t) - \beta I^*(t) & 0 & 0 \\ \beta I^*(t) & -d_1 & 0 \\ \alpha Z^*(t) & 0 & -d_2 \end{pmatrix} \emptyset(t),$$

Where $\emptyset(0) = I_3$ is the identity matrix. Hence the fundamental solution matrix is

$$\emptyset(t) = \begin{pmatrix} e^{\int_0^T [1-\alpha Z^*(t)-\beta I^*(t)]dt} & 0 & 0 \\ * & e^{-d_1T} & 0 \\ * & 0 & e^{-d_2T} \end{pmatrix}$$

Also, the fourth, fifth and sixth equations in (4) read as

$$\begin{pmatrix} p(nT^+) \\ q(nT^+) \\ r(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p(nT) \\ q(nT) \\ r(nT) \end{pmatrix}$$

Hence, if all eigen values of

$$M=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\emptyset(T)$$

have absolute values less than 1, then the periodic solution $(0, I^*(t), Z^*(t))$ is locally asymptotically stable. The eigen values of M are

$\lambda_1 = e^{-d_2T} < 1, \lambda_2 = e^{-d_1T} < 1, \quad \lambda_3 = e^{\int_0^T [1-\alpha Z^*(t)-\beta I^*(t)]dt}$.

It follows that $|\lambda_3| < 1$ if and only if $T < \left(\frac{\alpha v}{d_2} + \frac{\beta u}{d_1}\right)$ holds. Thus the Floquet theory of impulsive differential equations, in this situation, implies that the susceptible pest-eradication periodic solution $(0, I^*(t), Z^*(t))$ is locally asymptotically stable. The proof is complete.

Theorem 5.2 If $T < \left(\frac{\alpha v}{d_2} + \frac{\beta u}{d_1}\right)$, then the periodic solution $(0, I^*(t), Z^*(t))$ is globally asymtotically stable for the system (1).

Proof: By given condition and Theorem 5.1, it is easy to know that $(0, I^*(t), Z^*(t))$ is locally asymptotically stable. Therefore, we only need to prove its global attraction.

Since $T < \left(\frac{\alpha v}{d_2} + \frac{\beta u}{d_1}\right)$, we can choose a $\epsilon_1$small enough such that

$$\int_0^T [1 - \alpha\,(Z^*(t) - \epsilon_1) - \beta(I^*(t) - \epsilon_1)]dt = \sigma < 0.$$

Besides, we have

$$I'(t) = \frac{\beta S(t)I(t)}{1 + yS(t)} - d_1I(t) \geq -d_1I(t).$$

From Lemmas 3.2 and 4.3, there exists a n_1 such that for

$$I(t) \geq I'(t) - \epsilon_1, \text{ for } t \geq n_1T,$$

Similarly, there exists a n_2 $(n_2 > n_1)$ such that

$$Z(t) \geq Z^*(t) - \epsilon_1, \text{ for } t \geq n_2T.$$

Thus, for $t \geq n_2T$, we have

$$\begin{aligned} S'(t) &= S(t)(1 - S(t)) - \frac{\alpha S(t)\,Z(t)}{1 + xS(t)} - \frac{\beta S(t)I(t)}{1 + yS(t)} \\ &\leq S(t) - \frac{\alpha S(t)\,Z(t)}{1 + xS(t)} - \frac{\beta S(t)I(t)}{1 + yS(t)} \\ &\leq S(t)\left(1 - \frac{\alpha(Z^*(t)-\epsilon_1)}{1+xS(t)} - \frac{\beta(I'(t)-\epsilon_1)}{1+yS(t)}\right). \end{aligned}$$

From the above inequality, we get

$$S(t) \leq S(n_2T)e^{\int_{n_2T}^t \left[1-\frac{\alpha(Z^*(t)-\epsilon_1)}{1+xS(t)}-\frac{\beta(I'(t)-\epsilon_1)}{1+yS(t)}\right]dt} \leq S(n_2T)e^{k\sigma}.$$

Where $t \in ((n_2 + k)T, (n_2 + k + 1)T], k \in Z_+$. Since $\sigma < 0$, we can easily see that $S(t) \to 0$ as $k \to +\infty$.

Thus for arbitrary positive constant ϵ_2 small enough, there exist n_3 $(n_3 > n_2)$ such that

$S(t) < \epsilon_2$ for all $t \geq n_3 T$. From which we get
$$I'(t) = \beta S(t)I(t) - d_1 I(t) \leq (\beta\epsilon_2 - d_1)\, I(t),$$
From Lemmas 3.2 and 4.3, there exists a n_4 $(n_4 > n_3)$ such that
$$I(t) \leq I_2^*(t) + \epsilon_1, \text{ for } t \geq n_4 T,$$
where $I_2^*(t) = \frac{ue^{-(d_1 - \propto\epsilon_2)(t-kT)}}{1-e^{-(d_1 - \propto\epsilon_2)T}}$, for $t \in (kT,(k+1)T], k \in Z_+$.

By similarly argument, there exists a n_5 $(n_5 > n_4)$ such that
$$Z(t) \leq Z_2^*(t) + \epsilon_1, \text{ for } t \geq n_5 T,$$
where $Z_2^*(t) = \frac{ve^{-(d_2 - \beta\epsilon_2)(t-kT)}}{1-e^{-(d_2 - \beta\epsilon_2)T}}$, for $t \in (kT,(k+1)T], k \in Z_+$.

Note that ϵ_1, ϵ_2 are positive constants small enough and $I_2^*(t) \to I^*(t)$, $Z_2^*(t) \to Z^*(t)$ as $t \to \infty$. Therefore, the periodic solution $(0, I^*(t), Z^*(t))$ is globally asymtotically stable.

VI. REFERENCES

[1] DD Bainov and PS Simeonov (1993). Impulsive differential equations: periodic solution and applications. London: Longman.

[2] P Debach and D Rosen (1991). Biological control by natural enemies. 2nd ed. Cambridge: Cambridge University Press.

[3] B Goh (1980). Management and analysis of biological populations. Amsterdam, Oxxford, New york: Elsevier Scientific Publishing Company.

[4] Y Ding, S Gao, Y Liu and Lan (2010). A pest management epidemic model with time delay and stage structure, Appl. Math. 1 pp. 215-221.

[5] V Lakshmikantham, DD Bainov and PS Simeonov (1989). Theory of impulsive differential equations. Singapore: world Scientific.

[6] R Shi and L Chen (2010). An impulsive predator-prey model with disese in the prey for integrated pest management, Commun Nonlinear Sci Numer Simulat, pp. 421-429.

[7] S Zhang and L Chen (2005). A Holling II functional response food chain model with impulsive perturbations, Chaos Solitons and Fractals 24, pp. 1269-1278.

[8] HJ Freedman (1976). Graphical stability, enrichment and pest control by a natural enemy, Math Biosci, 31 pp. 207-25.

[9] ML Luff (1983). The potential of predators for pest control, Agri Ecos Environ, 10 pp. 159-81.

[10] JC Van Lenteren (2000). Measures of success in biological control of anthropoids by augmentation of natural enemies, In: Wratten S, Gurr G, editors, Measures of success in biological control, Dordrecht; Lluwer Academic Publishers, pp. 77-89.

[11] H Zhang, J Jiao, and L Chen (2007). Pest management through con-tinuous and impulsive control strategies. Biosystems 90, pp. 350–361.