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Inverse Burr Type X Distribution

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Abstract: The Burr type X distribution is considered as a base line model to get its inverse model. The distribution characteristics of the model along with its graphical shapes are presented. Maximum likelihood estimation of its parameters is derived and the results are illustrate with a live example. The fitness of the model to the data is also established.

Keywords: Inverse Transformation, MLE, Coefficient of Correlation, Asymptotic Variance

I. INTRODUCTION

If, X is a positive valued continuous random variable with cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x) with their inherent parameters, then the distribution of the random variable $Y = \frac{1}{X}$ is called the inverse distribution of the basic random variable. The cdf and pdf of the inverse distribution in terms of those of the baseline distribution $F(\cdot)$ are given by

$$G(\cdot) = 1 - F\left(\frac{1}{y}\right) \text{ .... } (1.1)$$

$$g(\cdot) = G'(y) = \frac{d}{dy}[G(y)] \text{ .... } (1.2)$$

If X follows exponential, then the distribution of Y is called inverse exponential, if X follows Rayleigh, then the distribution of Y is called Inverse Rayleigh, if X follows Gamma distribution then the distribution of Y is called Inverse Gamma distribution and so on.

Twelve types of families of distributions are suggested by Burr (1942). Among these twelve types of models, Burr type X distribution is a popular model. The inverse notion of Burr type X distribution is little known in the literature. We propose the inverse Burr type X distribution as a topic of study in this paper to the extent possible. The distributional characteristics and groups of inverse Burr type X distribution are presented in Section – 2. Maximum likelihood estimation and its illustration to a fitted live data are given in Section – 3.

II. INVERSE BURR TYPE X DISTRIBUTION

We know that a single parameter Burr type X distribution has the following density function and distribution function.

$$f(x;k) = 2kxe^{-x^2}(1-e^{-x^2})^{k-1}; \quad x>0, k>0 \text{ ... } (2.1)$$

$$F(x;k) = (1-e^{-x^2})^k; \; x>0, \; k>0. \text{ .... } (2.2)$$

If a new random variable Y is defined as $Y = \frac{1}{X}$ where X follows a Burr type X distribution, then using equations (1.1) and (1.2) with (2.1) and (2.2) the following are the basic distributional characteristics of the distribution of Y.

Probability density function (pdf):

$$g(y) = \frac{2k\,e^{-1/y^2}}{y^3}\left(1-e^{-1/y^2}\right)^{(k-1)}; k>0, y>0. \qquad (2.3)$$

Cumulative distribution function (cdf):

$$G(y) = 1-\left(1-e^{-1/y^2}\right)^{k}. \qquad (2.4)$$

Reliability function:

$$R(y) = \left(1-e^{-1/y^2}\right)^{k}. \qquad (2.5)$$

Hazard function:

$$h(y) = \frac{2k}{y^3\left(e^{1/y^2}-1\right)}. \qquad (2.6)$$

Mean Residual Life (MRL):

$$MRL = m(t) = \frac{2k}{\left(1-e^{-1/t^2}\right)^k}\int_t^{\infty}\frac{\left(1-e^{-1/u^2}\right)^k}{u^2\left(e^{1/u^2}-1\right)}du - t \qquad (2.7)$$

Mean Waiting Time (MWT) (Felipe *et al.*2011): In a given time interval (0,t) if it is known that the failure occurred somewhere in this interval, the lapse of time from the instant of failure to t is known as inactive period. This lapse of time is a random variable. The average value of this random variable is called Mean inactive time also usually called Mean Waiting Time. It is given by

$$MWT = w(t) = t-\int_0^t u\frac{f_T(u;k)}{F_T(t;k)}du \qquad (2.8)$$

$$MWT = w(t) = t-2k\int_0^t\frac{e^{-1/u^2}\left(1-e^{-1/u^2}\right)^{(k-1)}}{u^2}du \qquad (2.9)$$

The graphs of the frequency curves and hazard curves are given in the following figures for selected values of k. If k=1 Burr type X distribution becomes the well-known Rayleigh distribution and hence G(y) becomes the cdf of Inverse Rayleigh distribution. Our interest is for values of k different from one. We have chosen the values of k as $\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{2}{3}$, 1.5,2,2.5,3. The graphs of hazard functions show that this distribution has the phenomenon of an inverted bath tub shape.

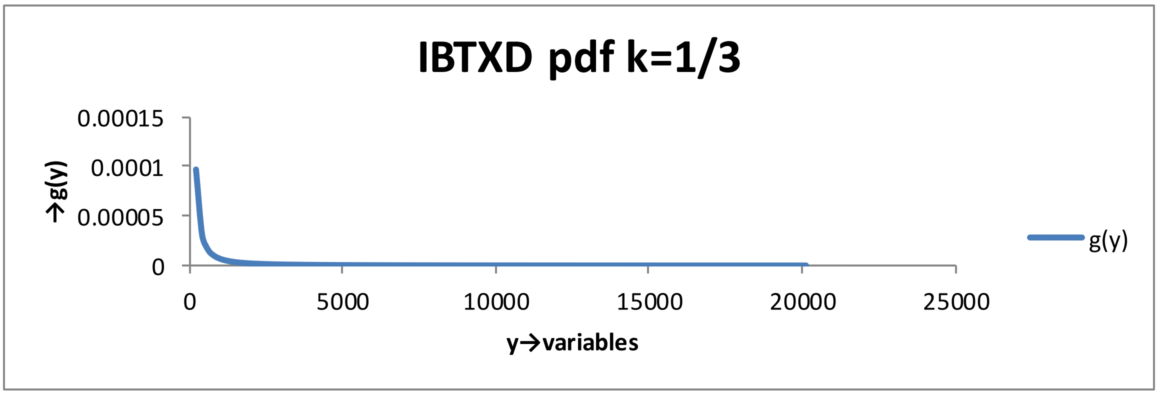
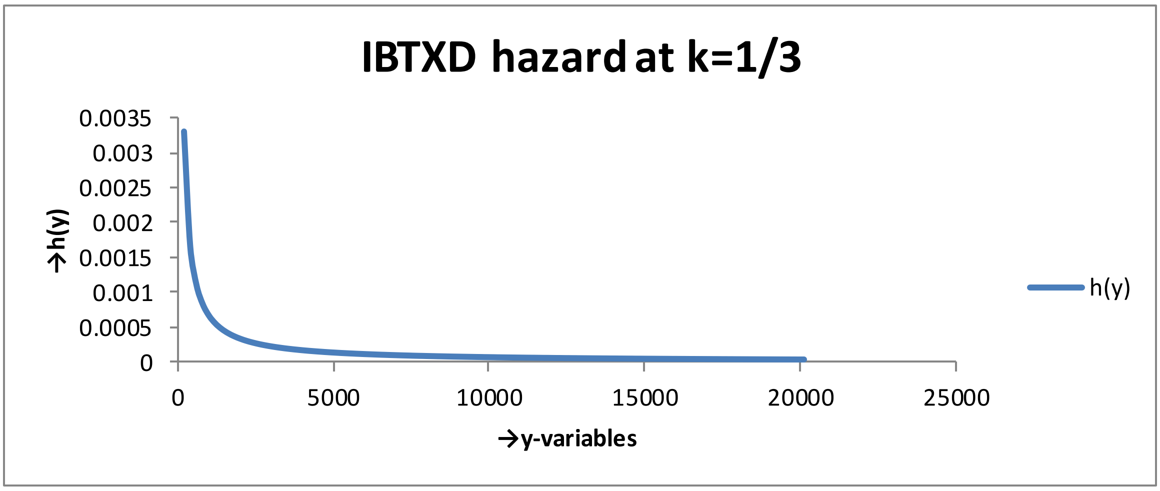


Figure – 2.1



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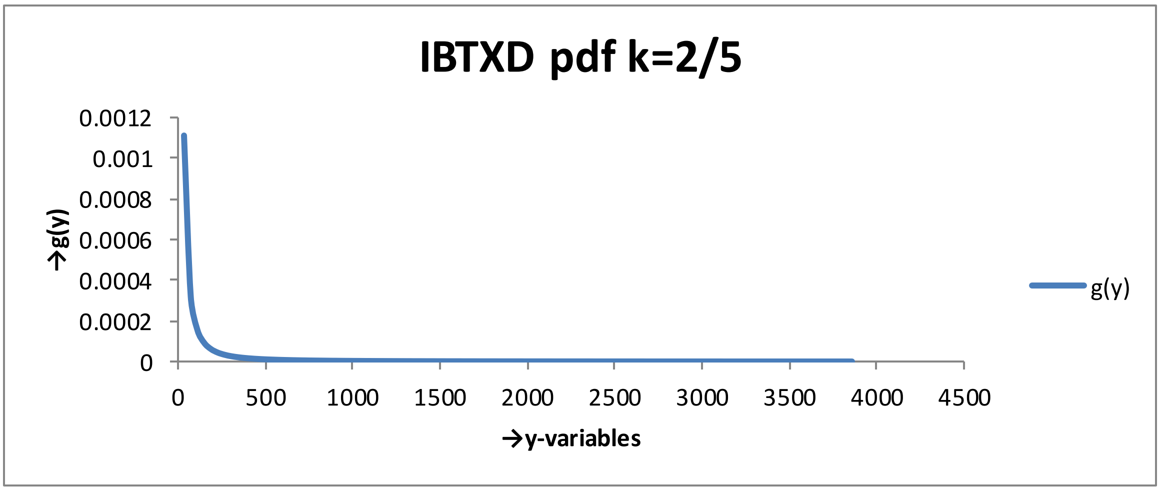

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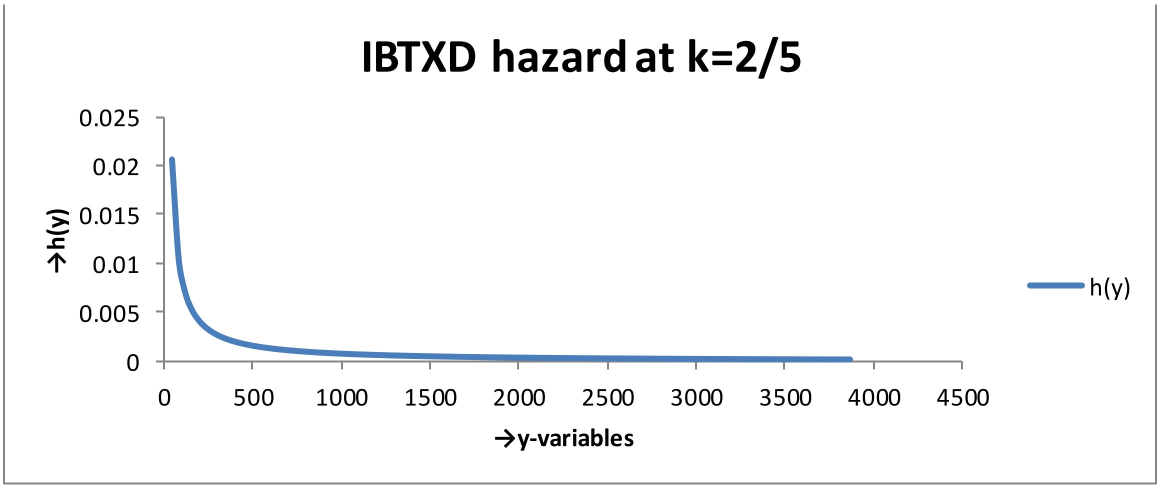

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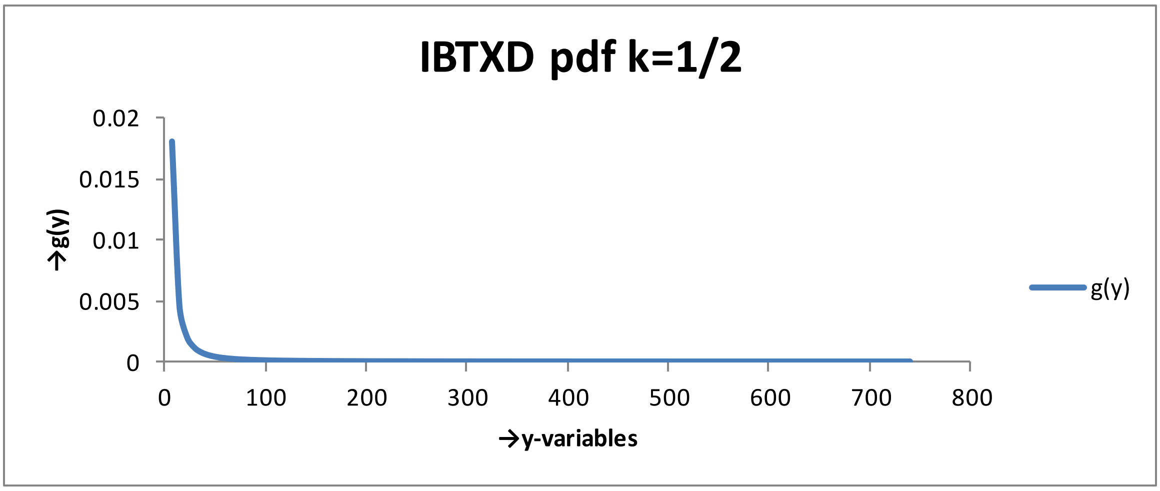

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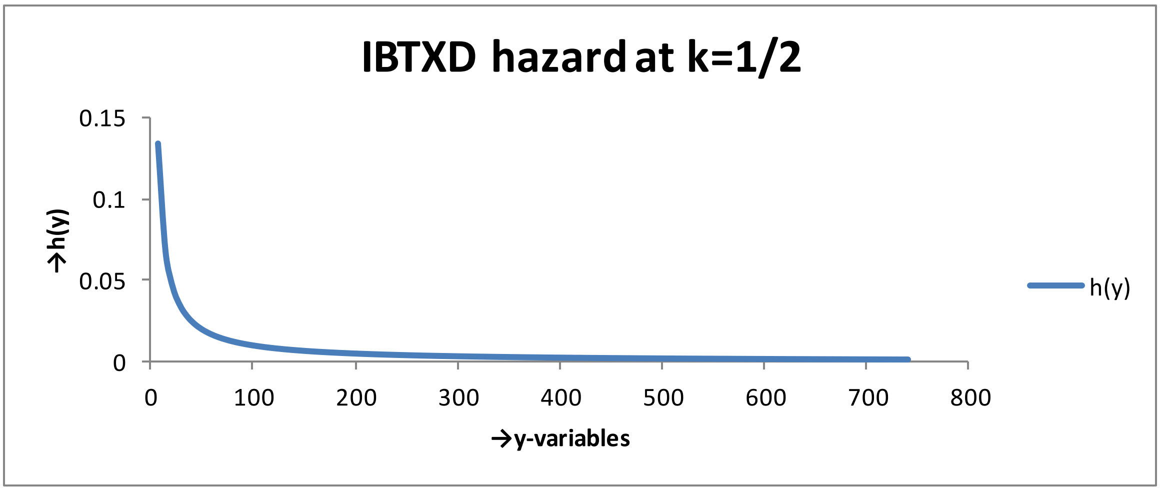

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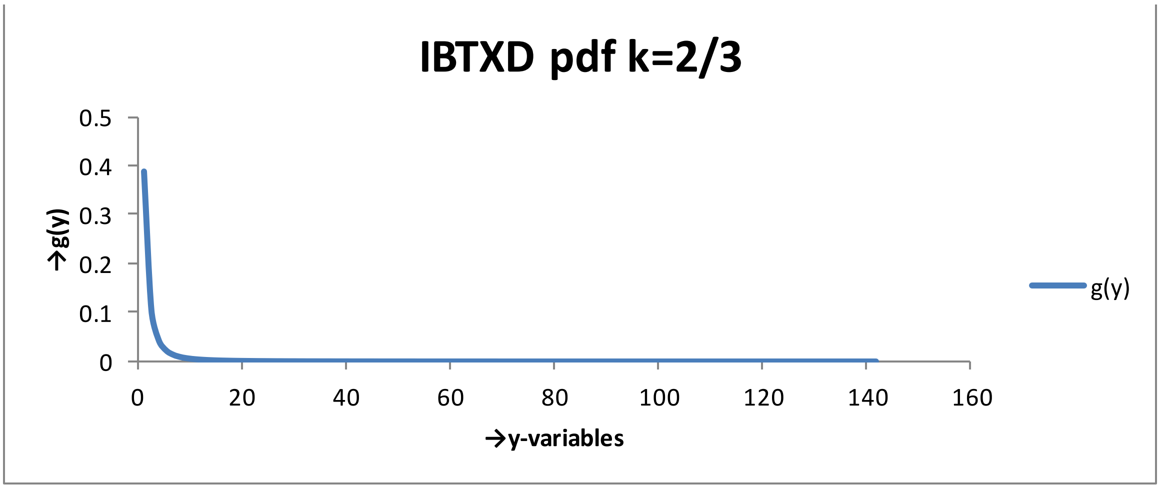

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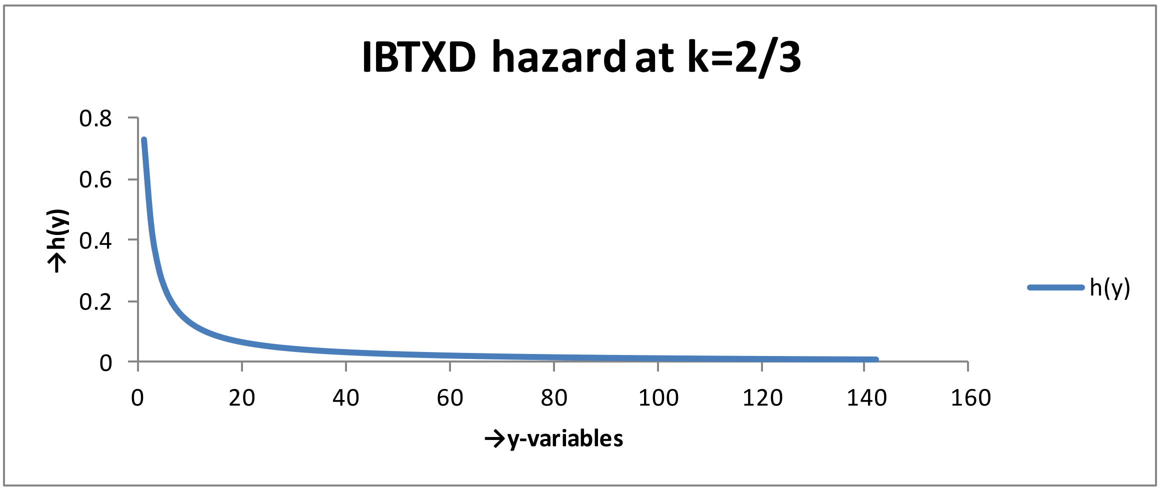

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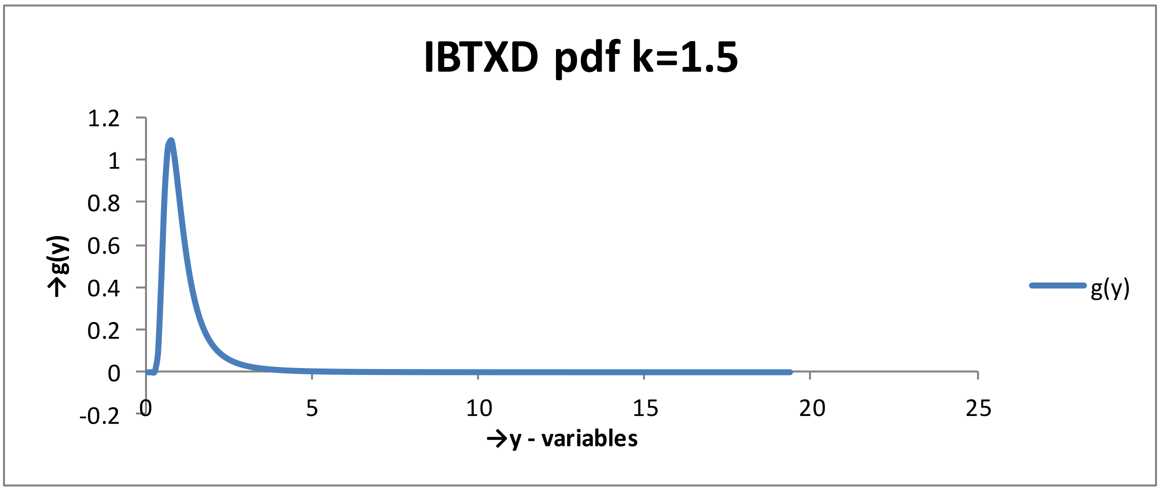

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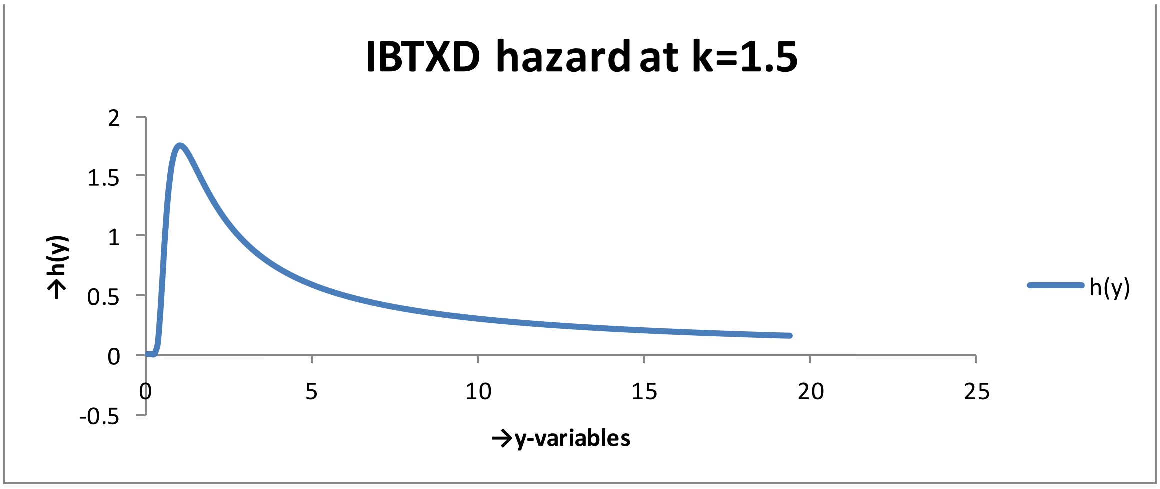

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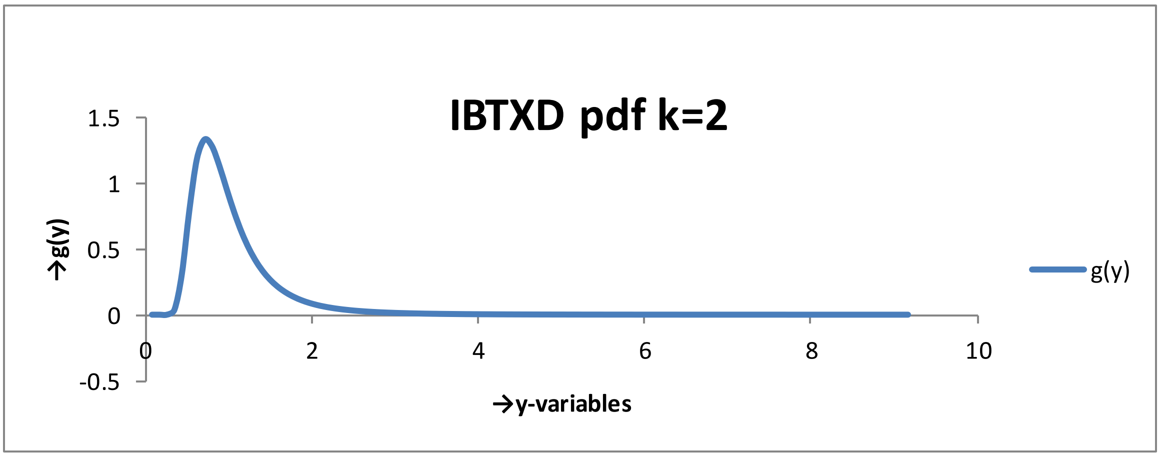

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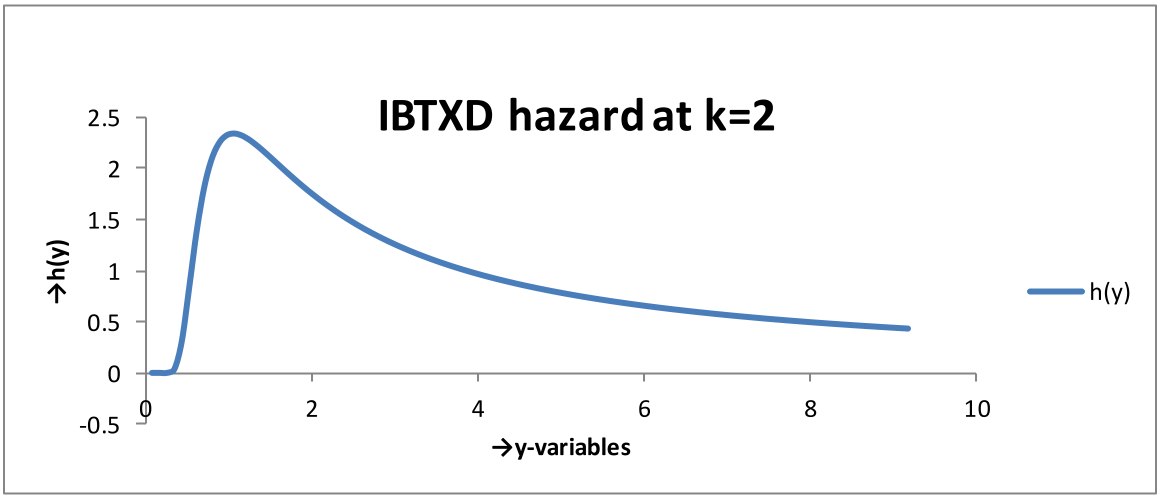

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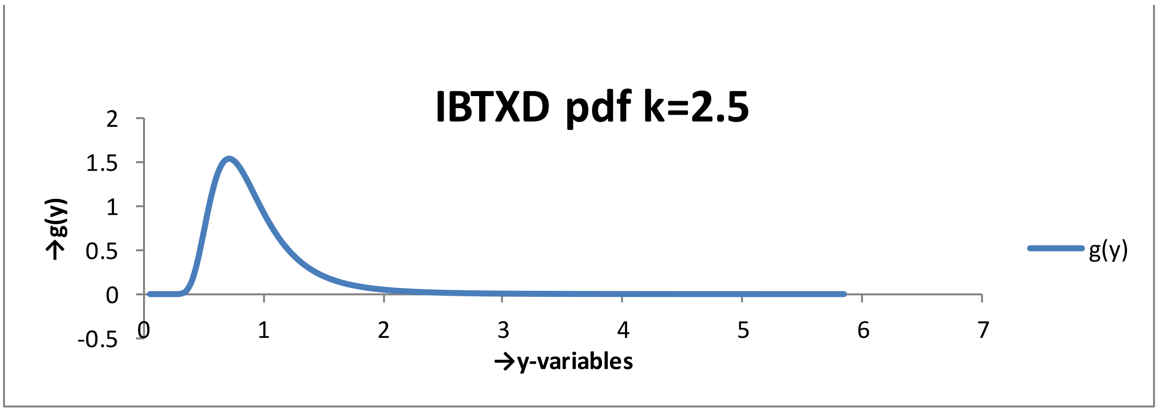

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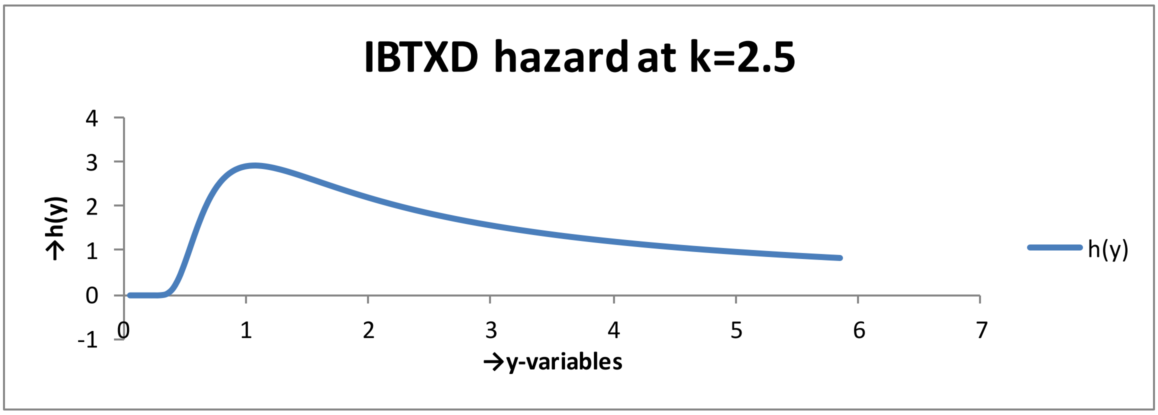

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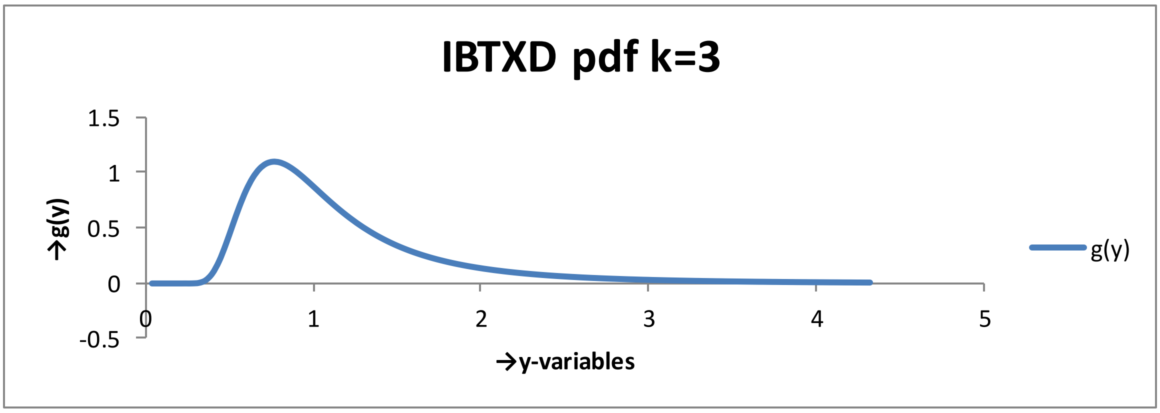

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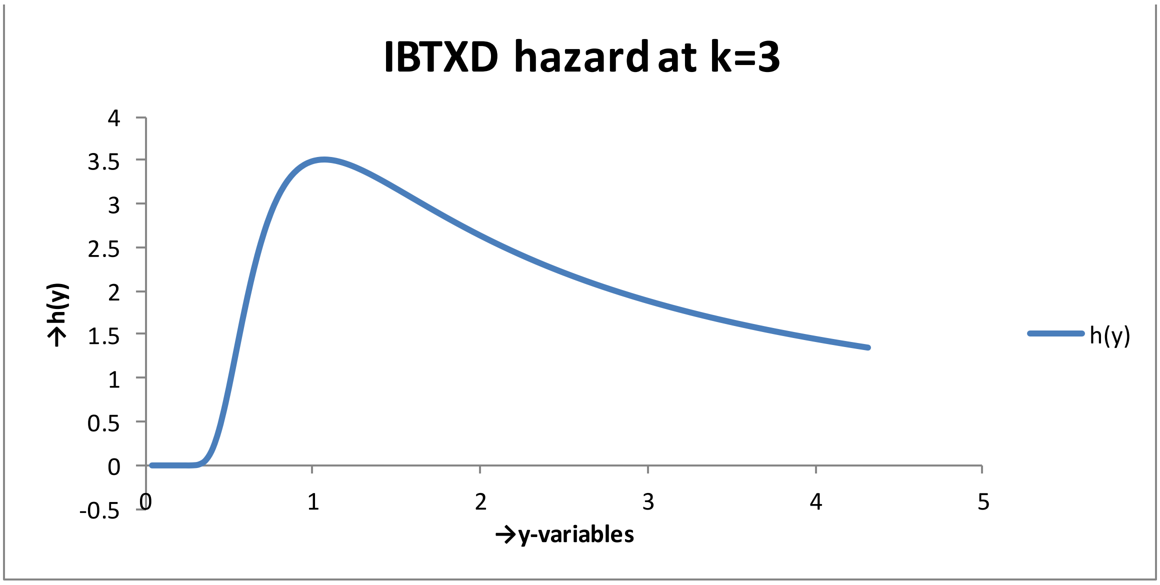

Figure – 2.16

Quantiles, Skewness and Kurtosis:

The p^{th} quantile of inverse Burr type X distribution is given by the solution of the equation $G(y) = p$ and is given by

$$y = \sqrt{\frac{-1}{Ln\left[1-(1-p)^{1/k}\right]}} \quad \text{.......} \quad (2.10)$$

Substituting $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ in succession we get the first quartile, median and third quartile. As these depend on the shape parameter k, for the chosen values of k these are as follows:

Table 1: Quantiles

Quartile / k	Q_1 (0.25)	Median(Q_2) (0.5)	Q_3 (0.75)
1/3	1.350901	2.736581	7.968606
2/5	1.22375	2.267301	5.61225
½	1.099845	1.864419	3.936321
2/3	0.976622	1.513997	2.736581
1.5	0.756854	1.00294	1.406374
2.0	0.705327	0.902423	1.201122
2.5	0.671281	0.839704	1.082023
3.0	0.646566	0.795954	1.00294

Bowley's coefficient of skewness, Kelly's coefficient of skewness and Moors' coefficient of kurtosis (Moors', 1998) are given by

$$\text{Bowley's Coeffiecient of Skewness} = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}, \quad \text{.......} \quad (2.11)$$

$$\text{Kelly's Coefficient of Skewness} = \frac{P_{90} - 2P_{50} + P_{10}}{P_{90} - P_{10}}, \quad \text{.......} \quad (2.12)$$

$$\text{Moors' Coefficient of Kurtosis} = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}. \quad \text{.......} \quad (2.13)$$

For chosen values of k these coefficients are given below.

Table 2: Coefficients of Skewness and Kurtosis

k	Bowely's Coefficient of Skewness	Kelly's Coefficient of Skewness	Moors' Coefficient of Kurtosis
1/3	0.58122	0.878892	2.626243
2/5	0.524416	0.829947	2.130335
1/2	0.460899	0.763356	1.695482
2/3	0.389332	0.673549	1.306793
1.5	0.242252	0.446229	0.710842
2.0	0.204931	0.381485	0.587236
2.5	0.179907	0.336942	0.508578
3.0	0.161623	0.303915	0.452927

Inverse Burr type X distribution is a positively skewed and platy kurtic distribution.

Mode:

It can be seen that the probability density function is a decreasing function of x, for $k \leq \frac{2}{3}$. For $k > \frac{2}{3}$, the mode of the inverse Burr type X distribution is solution of the equations $g'(y) = 0$ and $g''(y) < 0$. Differentiating $g(y)$ given in equation (3.3) with respect to 'y' and after some simplification we get the mode of the distribution as solution of the equation

$$\left(3 - \frac{2}{y^2}\right) - e^{-1/y^2}\left(3 - \frac{2k}{y^2}\right) = 0. \quad (2.14)$$

Mean:

The expected value of inverse Burr type X distribution is given by

$\int_0^\infty y g(y) dy = \int_0^\infty R(y) dy.$

$$\int_0^\infty y \frac{2k e^{-1/y^2}}{y^3}\left(1 - e^{-1/y^2}\right)^{(k-1)} dy = \int_0^\infty \left(1 - e^{-1/y^2}\right)^k dy \quad (2.15)$$

This integral has to be evaluated with the help of numerical integration only.

III. ESTIMATION

The inverse Burr type X distribution given by (2.3) is generally known as a single parameter inverse Burr type X distribution. Its parametric estimation by maximum likelihood method based on a complete random sample of size n is the solution of the corresponding log likelihood equation given by

$$\frac{\partial logL}{\partial k} = 0 \Rightarrow \frac{n}{k} + \sum_{i=1}^{n}\left[0 - 0 + Ln\left(1 - e^{-1/y_i^2}\right)\right] = 0. \quad (3.1)$$

Therefore, MLE of k is

$$\hat{k} = \frac{-n}{\sum_{i=1}^{n} Ln\left(1 - e^{-1/y_i^2}\right)}. \quad (3.2)$$

The asymptotic variance of $\hat{k}$ is the reciprocal of expectation of negative of $\frac{\partial^2 logL}{\partial k^2}$ and is given by

$$asvar\left(\hat{k}\right) = \frac{k^2}{n}. \quad (3.3)$$

If we are given a failure censored sample say $y_1 < y_2 < \cdots < y_r$ from out of a planned complete random sample of size n with the highest (n-r) observations censored, we proceed for the maximum likelihood estimation of k as follows. The likelihood is given by

$$L \propto \prod_{i=1}^{r} \frac{2k e^{-1/y_i^2}}{y_i^3}\left(1 - e^{-1/y_i^2}\right)^{(k-1)}\left(1 - e^{-1/y_r^2}\right)^{(n-r)}. \quad (3.4)$$

$$\frac{\partial logL}{\partial k} = 0 \Rightarrow \frac{r}{k} + \sum_{i=1}^{r} log\left(1 - e^{-1/y_i^2}\right) + (n-r) log\left(1 - e^{-1/y_r^2}\right) = 0. \quad (3.5)$$

The MLE of k from the failure censored sample is given by

$$\hat{k}_r = \frac{-r}{\sum_{i=1}^{r} log\left(1 - e^{-1/y_i^2}\right) + (n-r) log\left(1 - e^{-1/y_r^2}\right)} \quad (3.6)$$

The asymptotic variance of the MLE is the reciprocal of expectation of negative of $\frac{\partial^2 logL}{\partial k^2}$ to be obtained from (3.5) and is given by

$$asvar\ (\hat{k}_r) = \frac{k^2}{r}.\ ....\ (3.7)$$

It may be noted that the asymptotic variance of MLE of k from a failure censored sample depends only on the number of uncensored observations(r), but is not influenced by the size n of the originally planned sample. From the asymptotic properties of MLEs we know that

$$\hat{k} \sim N\left(k, \frac{k^2}{n}\right)$$

$$\hat{k}_r \sim N\left(k, \frac{k^2}{r}\right),$$

as the asymptotic sampling distributions of $\hat{k}, \hat{k}_r$ respectively. These would help us in developing asymptotic confidence intervals for k and tests of hypotheses on k.

Example

Nigm and Hamdy (1987) presented length of time in years for which a business operates until failure. It is considered that only the first 10 lifetimes of a random sample of 15 businesses are available. The data set is the following:

1.01,1.05,1.08,1.14,1.28,1.30,1.33,1.43,1.59,1.62. The uncensored observations are r=10 and the size of the originally planned sample n=15.

These sample observations are considered as to have come from an inverse Burr type X distribution with the possible values of the shape parameter as $k = \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, 1.5, 2, 2.5, and\ 3.$ Let z_i be the solution of equation $G(y) = \frac{i}{n+1}, i = 1,2, \dots, r,$ where n=15, r=10, and $G(\cdot)$ be the cdf of inverse Burr type X distribution. That is, z_i is i^{th} quantile of inverse Burr type X distribution. If $x_1, x_2, \dots, x_r$ denote the given censored sample with n=15, r=10 the coefficient of correlation between z_i and x_i for various values of the shape parameter k, may be taken as an indicator (the essence of well-known Q-Q Plot) of the fitness of the data to inverse Burr type X distribution. For the present example we have calculated these correlation coefficients with the above chosen values of k and are given below:

Table 3: Fitness of the Distribution - Correlation Coefficients

k	1/3	2/5	½	2/3	1.5	2.0	2.5	3.0
Correlation Coefficient	0.973087	0.979422	0.984279	0.987249	0.985673	0.983619	0.981844	0.980338

Among these, the correlation coefficient for $k = \frac{2}{3}$ is the maximum, though all of them are more than 0.97. That is, inverse Burr type X distribution with $k = \frac{2}{3}$ is the most suitable model to the sample data' though at all the remaining values of k, the coefficient of correlation is quite significant. Using the sample data we can get the MLE of k in an inverse Burr type X model given by the formula (3.6) namely,

$$\hat{k}_r = \frac{-r}{\sum_{i=1}^{r} log\left(1-e^{-1/y_i^2}\right) + (n-r) log\left(1-e^{-1/y_r^2}\right)}.$$

Substituting the given censored sample in the above formula we get the MLE of k as $\hat{k}_r = 0.737703$.

$asvar\ (\hat{k}_r) = \frac{k^2}{r}$

$asvar\ (\hat{k}_r) = 0.04444$

The 95% confidence interval for $\hat{k}_r$ is given by $k \pm 1.96\frac{k}{\sqrt{r}} = (0.253462, 1.079871)$.

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